FM for HOAS

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For this talk fix some countably infinite **set of atoms** $a, b, c, \ldots \in \mathbb{A}$. Let a **swapping** be a function $(a\ b) : \mathbb{A} \longrightarrow \mathbb{A}$ defined by

(1)
$$
\begin{aligned}
(b\ a)a &\stackrel{\text{def}}{=} b \\
(b\ a)b &\stackrel{\text{def}}{=} a \\
(b\ a)n &\stackrel{\text{def}}{=} n \qquad n \neq a, b.
\end{aligned}
$$

A word on notation: we shall tend to treat $a, b, c \in \mathbb{A}$ as constants in the sense that if $a \not\equiv b$ (syntactic equality on the page) then we may assume $a \neq b \in \mathbb{A}$ (semantic equality between atoms), unless otherwise stated.

Let $\pi, \pi', \kappa \in P_{\mathbb{A}}$ be the **set of finite permutations of atoms**, thus the subset of $\mathbb{A}^{\mathbb{A}}$ inductively generated by the swappings $(a\ b)$ and **Id** the identity on $\mathbb{A}$. This is a group with unit **Id** under functional composition $\circ$.

Let the category of **Nominal Sets** have objects sets with $P_{\mathbb{A}}$ action—

(2) $\forall \pi, \pi', x.\ \pi \cdot (\pi' \cdot x) = \pi \circ \pi' \cdot x$ and $\mathbf{Id} \cdot x = x$

(the standard rules for a permutation action)—and **finite support**

(3) $$\forall x \in X.\ \text{И} a, b.\ (a\ b) \cdot x = x.$$

Write $\mathcal{P}_{fin}(\mathbb{A})$ for the set of finite sets of atoms. Write $\text{И} a.\ \Phi(a)$ for $\exists S \in \mathcal{P}_{fin}(\mathbb{A}).\ \forall a \notin S.\ \Phi(a)$. Then (3) above means

(4) $$\forall x \in X.\ \exists S \in \mathcal{P}_{fin}(\mathbb{A}).\ \forall a, b \notin S.\ (a\ b) \cdot x = x.$$

Arrows in NOM are functions $f \in X \to Y$ which commute with the permutation action:

(5) $$\forall a, b \in \mathbb{A}, x \in X.\ f((a\,b) \cdot x) = (a\,b) \cdot f(x).$$

NOM is equivalent to the **Schanuel topos**, a boolean topos. NOM is a natural category of **equivariant** FM **sets**, a set theory similar to ZFA but with an extra axiom corresponding to (3). Thus in NOM we have a language of arrows very similar to a classic set theory and this (theoretical) fact gives us good programming and logic, see for example $\mathrm{FreshML}$ and Nominal Logic. They gave me a thesis for that in 2001.

[Peculiar warbing sound-effects] We now transport to a different world. . .

Weak HOAS

Model variable binding by meta-level binding. Concretely that means the abstraction type-former is $\mathbb{A} \to -$. Thus untyped λ-terms are

$$\Lambda \cong \mathbb{V} + \Lambda \times \Lambda + \Lambda^{\mathbb{A}}.$$

This kind of thing turns up all the time, all over the place; in $Set^{\mathbb{F}}$, in COQ, in the Theory of Contexts. It's convenient: $\times$ and $\to$ are everywhere.

Weak HOAS

It does *not* turn up in FM, there the abstraction type-former is $[\mathbb{A}]-$ which is a kind of tensor function space $\mathbb{A} \multimap -$, a right adjoint to $\mathbb{A} \otimes -$.

Advantages of FM: better arrows. Disadvantages: $\otimes$ and $\multimap$ exist only in NOM and we *still* lack a categorical axiomatisation.

Advantages of HOAS: already mentioned. Disadvantages: $\mathbb{A} \longrightarrow X$ tends to explode. Something has to break. In $Set^{\mathbb{F}}$ the topos is not boolean (bad arrows), in the Theory of Contexts they lose unique choice and thus the ability to turn graphs of functions into functions (more bad arrows, really).

Question: can we do FM to weak HOAS? This is no earth-shattering question, but it bugged me enough that I cooked up an answer.

Let a **renaming** be a function $[a \leftarrow\!\dashv b] : \mathbb{A} \to \mathbb{A}$ defined by

(6)
$$[a \leftarrow\!\dashv b]b \stackrel{\text{def}}{=} a$$
$$[a \leftarrow\!\dashv b]n \stackrel{\text{def}}{=} n \qquad n \neq a$$

and write $Sub_{\mathbb{A}}$ for **atom-for-atom substitutions on** $\mathbb{A}$; the monoid generated by the $[a \leftarrow\!\dashv b]$ with functional composition $\circ$ as the monoid action. Write **Id** for the identity function on $\mathbb{A}$, which is the unit of the monoid $(Sub_{\mathbb{A}}, \circ)$.

A **Substitution Algebra** is a set X with an $Sub_{\mathbb{A}}$ monoid action ${}^{\sigma}$ (we may drop it)

(7) $$\forall \sigma, \sigma', x.\ \sigma^{\sigma}({\sigma'}^{\sigma} x) = \sigma \circ {\sigma'}^{\sigma} x \quad \text{and} \quad \mathbf{Id}^{\sigma} x = x$$

along with a mysterious **"consistency condition"**

(8) $$\forall x \in X.\ \forall a, b, a', b'.\ b \neq b' \implies [a \leftarrow\!\dashv b][a' \leftarrow\!\dashv b']x = [a' \leftarrow\!\dashv b']x \implies [a \leftarrow\!\dashv b]x = x$$

and a **"finite support property"**

(9) $$\forall x \in X.\ \text{И} b.\ \forall a.\ [a \leftarrow\!\dashv b]x = x.$$

Arrows of SUB

Arrows of SUB are maps $f : X \to Y$ which commute with the renaming action:

(10) $$\forall a, b \in \mathbb{A}, x \in X.\ f([a \hookleftarrow b]x) = [a \hookleftarrow b]f(x).$$

So given $a \in \mathbb{A}$ and $x \in X \in$ SUB there is an obvious abstraction of x by a given by $\lambda b.[b \hookleftarrow a]x$. As we shall see, there are no exotic terms and SUB will be a 'category for weak HOAS'.

SUB has a notion of support

In NOM we have a notion of **support**; for any $x \in X \in$ NOM there is a set $S(x) = \{a \mid a\#x\}$ where $a\#x$ read **"a fresh for x"** when $\mathsf{N}b.\ (b\ a)x = x$. This is an advantage of NOM over $Set^{\mathbb{F}}$ because you don't have to index all your calculations with a set of 'known names' in the context, e.g. in order to choose a fresh one.

SUB also enjoys a notion of support. $a\#x$ when $\mathsf{N}b.\ [b \hookleftarrow a]x = x$. Support satisfies the following nice property:

Theorem 1. $S(\alpha^{\sigma} x) = \alpha' S(x)$

where $\alpha' U$ denotes the pointwise action. So there is a good sense in which x 'contains' atoms and $[a \hookleftarrow b]$ 'renames' them. We shall see more of this in the lemma of the next slide.

SUB has a notion of simultaneous substitution

Note that the monoid $Sub_{\mathbb{A}}$ lacks the simultaneous substitution $[a \leftarrow\!\dashv b, b \leftarrow\!\dashv a]$, otherwise known as the swapping $(a\ b)$. However we can simulate this action by

(11) $$(a\ b)_c x \stackrel{\text{def}}{=} [a{\leftarrow\!\dashv}c][b{\leftarrow\!\dashv}a][c{\leftarrow\!\dashv}b]$$

for any $c\#x$. The correctness of this simulation follows from the following lemma:

Lemma 2. *If U supports x and $\alpha, \beta \in Sub_{\mathbb{A}}$ are such that $\forall u \in U.\ \alpha(u) = \beta(u)$—write this condition henceforth as $\alpha|_U = \beta|_U$—then $\alpha^\sigma x = \beta^\sigma x$.*

As a corollary, any Substitution Algebra is a Nominal Set, and in fact SUB is a category of algebras over NOM.

SUB is Cartesian Closed

$\times$ is product on underlying sets. The unit is 1 the one element set (and terminal object). The exponential $Y^X \in$ SUB has underlying set

(12) $$\left\{ f : X \to Y \;\middle|\; \text{И}b.\, \forall a.\, \forall x.\, [a \leftarrow\!\!\dashv b](f(x)) = f([a \leftarrow\!\!\dashv b]x) \right\},$$

with action

(13) $$([a \leftarrow\!\!\dashv b]f)x \;\stackrel{\text{def}}{=}\; \iota z.\, \text{И}b'.\, z = [b \leftarrow\!\!\dashv b']\,[a \leftarrow\!\!\dashv b](f[b' \leftarrow\!\!\dashv b]x).$$

SUB and variable binding

Recall that SUB as a swapping action given by $(a\ b)x = (a\ b)_c x$ for $c\#x$. This gives a forgetful functor $\mathcal{U}$: SUB $\longrightarrow$ NOM. Recall also forgetful functors to Set. Then

$$\mathcal{U}(X^{\mathbb{A}}) \cong \mathcal{U}([\mathbb{A}](\mathcal{U}X));$$

function abstraction $X^{\mathbb{A}}$ in SUB is, underlying-set-wise, FM abstraction $[\mathbb{A}]\mathcal{U}X$.

Theorem 3. *The endofunctor* $-^{\mathbb{A}}$: *SUB* $\longrightarrow$ *SUB commutes with limits, colimits, and function spaces.*

Thus

$$(X \times Y)^{\mathbb{A}} \cong X^{\mathbb{A}} \times Y^{\mathbb{A}}$$
$$(X + Y)^{\mathbb{A}} \cong X^{\mathbb{A}} + Y^{\mathbb{A}}$$
$$(Y^X)^{\mathbb{A}} \cong (Y^{\mathbb{A}})^{(X^{\mathbb{A}})},$$

and

$$\Lambda \cong \mathbb{A} + \Lambda \times \Lambda + \Lambda^{\mathbb{A}}$$

in SUB is naturally λ-terms up to α-equivalence. And being inductively defined it comes with inductive programming principles.

But. . .

SUB is not a topos

SUB is not a topos (proof omitted). Furthermore the map on underlying sets $= : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ is not an arrow in SUB because it does not commute with the renaming action:

$$[a \hookleftarrow b](a = b) = [a \hookleftarrow b]\bot = \bot$$
$$[a \hookleftarrow b](a = b) = ([a \hookleftarrow b]a = [a \hookleftarrow b]b) = (a = a) = \top.$$

SUB is not a good place to do logic in. It isn't such a hot place to program in either because of the theorem on the following slide. . .

Write $x\#y$ when $S(x) \cap S(y) = \emptyset$ (generalising previous notation).

Theorem 4. *For $f, g : Y^X \in$ SUB,*

$$\Big(\forall x \in X.\, x\#f, g \implies f(x) = g(x)\Big) \implies f = g. \tag{14}$$

Sketch proof. Suppose we assume the hypothesis of the implication above. If $b\#h$ then

$$[a{\leftarrow\!\dashv}b]h(x) = ([a{\leftarrow\!\dashv}b]h)([a{\leftarrow\!\dashv}b]x) = h([a{\leftarrow\!\dashv}b]x).$$

By assumption $f(x') = g(x')$ for $x' = [\vec{b}{\leftarrow\!\dashv}\vec{a}]x$ where $\vec{a}$ is $S(x)$ in order and $\vec{b}$ is a vector of fresh atoms. Then $[\vec{a}{\leftarrow\!\dashv}\vec{b}]f(x') = f(x)$ and similarly for g. □

What is SUB? A category of contexts

It seems to me that SUB is a ‘category of contexts’. Atoms a, b, c represent holes rather than variable names. Functions cannot actually examine the names of holes so, unlike in FM, passing their names around is pointless. Is there an application for which this restriction is a feature?

Contexts spring immediately to mind: we do not want a function in our universe taking $C[-_1, -_2]$ to $\top$ if $-_1 \equiv -_2$ and $\bot$ if $-_1 \not\equiv -_2$.

What is SUB? A category of algebras over NOM

Let $TX \in$ NOM be $(Sub_{\mathbb{A}} \times X)/\sim$, where $\sim$ is the equivalence relation generated by

$$\text{И}c.\ \langle \alpha \circ (a\ b)_c, x\rangle \sim \langle \alpha, (a\ b)x\rangle \qquad \text{and}$$
$$\alpha|_{S(x)} = \beta|_{S(x)} \Rightarrow \langle \alpha, x\rangle \sim \langle \beta, x\rangle.$$

Let $\text{NOM}^{\mathbb{T}}$ be the Eilenberg-Moore category of T-monad algebras and algebra maps between them. $\text{NOM}^{\mathbb{T}} \cong$ SUB.

Directions

SUB is very interesting; it's Cartesian Closed, complete, and co-complete. It has a notion of support (so our programs do not have to carry around a context of 'known names'). Abstraction $-^{\mathbb{A}}$ commutes with limits and colimits.

What else is it good for? I don't know. Ideas: application to contexts. If the arrows in SUB are too weak, *perhaps* we can treat it as a category of algebras over NOM and use those in NOM; we might find some existing theories of contexts can be viewed as an instance of that. SUB also indicates the kind of structure we might obtain by 'applying FM' to other notions of substitution.