Fresh Logic

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Fresh Logic and FM Sets

The И-quantifier is familiar from FM-sets. FM-sets is a first-order logic with constant symbol $\mathbb{A}$ and two binary predicates $=$ and ϵ, and some axioms. In that logic И is definable by

$$(1) \qquad \text{И}n.\, P \overset{\text{def}}{\iff} \exists L \in \mathcal{P}_{cofin}(\mathbb{A}).\, \forall n \in L.\, P.$$

Various properties of И are provable in this set theory, for example $\neg \text{И}n.\, P \Leftrightarrow \text{И}n.\, \neg P$ the some/any property of fresh names, so useful in programming and logic.

Various logics and programming languages have been translated into this theory and some are quite well-developed: FreshML and Cardelli-Caires spatial logics are two examples, also Bill Rounds last week. They all make heavy use of N.

Q. What is this quantifier?

QQ. What is that question?

AA. We want a natural-deduction system with a $\mathtt{NEW}$ quantifier along with a semantics $[\![\cdot]\!]$ in FM sets such that

$$[\![\mathtt{NEW}\,\mathtt{n}.\mathtt{P}]\!] \quad \text{is} \quad \mathsf{N}n.[\![\mathtt{P}]\!].$$

A completeness proof ties derivability in the formal logic to validity in the semantics. Proof-normalisation gives “good behaviour” (subformula property etc.) and also has some computational content.

A. One such system with semantics.

Hey. Call it Fresh Logic.

Application of Fresh Logic

A vanilla theory? Extend Fresh Logic with extra constants, axioms, and so on, and build logics; just like we do in First-Order Logic.

If we give ourselves a type of atoms and a type of π-calculus terms up to structural congruence, and extend with constructors appropriate to this domain, we would hope and expect to recover Spatial Logic.

I did this with Luis Caires recently, and given time I shall discuss the results.

Mathematical specification of problem

Dependent types? Write down a term-calculus. We should get a dependent type theory for FM.

Self-dual quantifiers? Call a quantifier ∇ self-dual when $\neg\nabla x.\phi \equiv \nabla x.\neg\phi$.

И is self-dual. So is the Tiu-Miller ∇. This appears to be new is logic, and I see this proof-theory for И as possibly an example of a wider logical phenomenon.

$$
\begin{gathered}
X ::= \mathtt{A} \mid \mathtt{C} \mid X \to X \mid \ldots \\
a \in \mathbb{A} \qquad x \in \mathbb{V} \qquad \pi ::= \mathbf{Id} \mid (a\ b) \circ \pi \\
t ::= a \mid \pi \cdot x \mid \mathsf{c}(ts) \\
P ::= p(ts) \mid P \wedge P \mid \neg P \mid \ldots \mid \forall x.\ P \mid \exists x.\ P \mid \text{И} n.\ P
\end{gathered} \tag{2}
$$

1. Sorts are X, $\mathtt{A}$ is the sort of atoms.
2. $a, b, n \ldots \in \mathbb{A}$ are atom constants.
3. $(a\ b)$ is a formal pair $\langle a \rangle b$ of a and b, which we call a swapping. π is a list of swappings (call **Id** the identity).
4. c is a term-constructor, p a predicate constant. ts in $\mathsf{c}(ts)$ and $p(ts)$ represents a list of t (I elide an arity system on c and p.
5. n in $\text{И} n.\ P$ is an atom constant, but bound.

Deduction rules for $\#$

(Axiom) $\Gamma, P \vdash P$

($\forall$I) $$\frac{\Gamma \vdash P}{\Gamma \vdash \forall x.\, P} \quad x \notin FV(\Gamma)$$

($\forall$E) $$\frac{\Gamma \vdash \forall x.\, P}{\Gamma \vdash P\{t/x\}} \quad x \notin FV(\Gamma, C)$$

($\#$I) $\Gamma \vdash a\#b$

($\#$E) $$\frac{\Gamma \vdash a\#a}{\Gamma \vdash \bot}$$

(So atoms $a, b, c, \ldots$ behave like constants, though we can also view them as given by an element of $\mathbb{A} \otimes \mathbb{A} \otimes \mathbb{A} \ldots$.)

Deduction rules for π

(πI) $$\frac{\Gamma \vdash P}{\Gamma \vdash \pi{\cdot}P}$$

(πdiff) $$\frac{\Gamma \vdash P\{\pi'{\cdot}t/x\} \quad \Gamma \vdash ds(\pi,\pi')\#t}{\Gamma \vdash P\{\pi{\cdot}t/x\}} \quad x \notin FV(t)$$

$$\pi{\cdot}a = \pi(a) \quad \pi{\cdot}(\kappa \cdot x) = \pi \circ \kappa \cdot x \quad \pi{\cdot}\mathsf{c}(ts) = \mathsf{c}(\pi{\cdot}ts)$$
$$\pi{\cdot}p(ts) = p(\pi{\cdot}ts) \quad \pi{\cdot}(P \wedge Q) = \pi{\cdot}P \wedge \pi{\cdot}Q \quad \ldots$$
$$\pi{\cdot}\forall x.\, P = \forall x.\, \pi{\cdot}P \quad \pi{\cdot}\exists x.\, P = \exists x.\, \pi{\cdot}P \quad \pi{\cdot}\mathsf{N}n.\, P = \mathsf{N}n.\, \pi{\cdot}P.$$

The two rules are not interdefinable:

$$\begin{aligned}\pi{\cdot}p(\kappa \cdot x, \kappa' \cdot x) &= p(\pi \circ \kappa \cdot x, \pi \circ \kappa' \cdot x)\\ &\neq p(\kappa \cdot x, \kappa' \cdot x)\{\pi \cdot x/x\}.\end{aligned}$$

Deduction rules for π

I am rather pleased by this definition of the π action. A more traditional one would be $\pi{\cdot}\forall x.\, P \equiv \forall x.\, \pi{\cdot}P\{\pi^{-1} \cdot x/x\}$ and similarly for $\exists$ and $\reflectbox{N}$.

Perhaps I should do some commutation cases on the board. And remind me to think about doing the essential case for the two $\reflectbox{N}$ rules:

Deduction rules for И

(ИI)
$$\frac{\Gamma \vdash P\{a/n\} \quad \Gamma \vdash a\#t_i \; (i = 1, \ldots, k)}{\Gamma \vdash \text{И}n.\, P}$$
$$P/n = P' \bullet_{y_i} (t_i)_1^k \quad a \notin \mathrm{A}(P')$$

(ИE)
$$\frac{\Gamma \vdash \text{И}n.\, P \quad \Gamma \vdash a\#t_i \; (i = 1, \ldots, k)}{\Gamma \vdash P\{a/n\}}$$
$$P/n = P' \bullet_{y_i} (t_i)_1^k \quad a \notin \mathrm{A}(P')$$

Deduction Rules for $\reflectbox{N}$

$P/n = P' \bullet_{y_i} (t_i)_1^k$ means that (for some y_1 to y_k not in $V(P)$) $P = P'\{t_i/y_i\}_1^k$, where this is capture-avoiding substitution, and $n \notin \mathrm{A}(t_i)$, and the t_i are maximal such. For example,

$$\forall x.\, p((d\,n)\cdot x, (d\,n)\circ(d\,d')\cdot z)\ /n = \\ \forall x.\, p((d\,n)\cdot x, (d\,n)\cdot y_1) \quad \bullet\ ((d\,d')\cdot z).$$

Spatial Logic has a corresponding notion of free term. At issue in ($\reflectbox{N}$I) and ($\reflectbox{N}$E) is a maximal set of free terms which do not mention n.

(New) $$\frac{\Gamma, a\#ts \ \vdash \ C}{\Gamma \ \vdash \ C} \qquad a \notin \mathrm{A}(\Gamma, C)$$

This rule tells us we can always generate a fresh name. We use it in conjunction with (ИI) or (ИE):

(3) $$\cfrac{\cfrac{\cfrac{}{a\#x \ \vdash \ a\#x}\text{(Axiom)}}{a\#x \ \vdash \ \text{И}n.\, n\#x}\text{(ИI)}}{\vdash \ \text{И}n.\, n\#x}\text{(New).}$$

Here $P/n = n\#y \bullet (x)$.

Example: $\text{И}b.\,(b\;a)\cdot x = x \;\vdash\; a\#x$

$$
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          \dfrac{}{b\#x, \text{И}b.\,(b\;a)\cdot x = x \;\vdash\; \text{И}b.\,(b\;a)\cdot x = x}\,(\text{Axiom})
        }{b\#x, \text{И}b.\,(b\;a)\cdot x = x \;\vdash\; (b\;a)\cdot x = x}\,(\text{ИE})
        \qquad
        \dfrac{}{b\#x, \text{И}b.\,(b\;a)\cdot x = x \;\vdash\; b\#x}\,(\text{Axiom})
      }{b\#x, \text{И}b.\,(b\;a)\cdot x = x \;\vdash\; b\#(b\;a)\cdot x}\,(=\text{E})
    }{b\#x, \text{И}b.\,(b\;a)\cdot x = x \;\vdash\; a\#(b\;a)\circ(b\;a)\cdot x}\,(\pi\text{I})\;(\pi = (b\;a))
  }{b\#x, \text{И}b.\,(b\;a)\cdot x = x \;\vdash\; a\#x}\,(\pi\text{diff})
}{\text{И}b.\,(b\;a)\cdot x = x \;\vdash\; a\#x}\,(\text{New})
$$

(Const$\mathbb{A}$) $$\frac{\bigwedge a \in \mathbb{A}.\, \Gamma\{a/x\} \vdash P\{a/x\}}{\Gamma\{t/x\} \vdash P\{t/x\}}$$

(Const$\mathbb{A}$) says the atoms-constants $a, b, c, \ldots$ are an exhaustive list. It gives us a lot of nice results:

1. $\text{И} n.\, p(n) \vdash \forall x.\, p(x)$ for $p : \mathtt{A}$ a predicate symbol.
2. $\vdash x \# y \iff x \neq y$.
3. $\vdash x =_{\mathtt{A}} y \vee x \neq_{\mathtt{A}} y$ (even in intuitionistic Fresh Logic).
4. $\vdash \mathtt{f}(x) = x$ for $\mathtt{f} : \mathtt{A} \longrightarrow \mathtt{A}$ a function symbol.

(Small) $$\frac{\bigwedge L \in \mathcal{P}_{cofin}(\mathbb{A}).\, \Gamma, L\#t \;\vdash\; P}{\Gamma \;\vdash\; P}$$

(Small) does not seem to imply such neat results. It and (Const$\mathbb{A}$) are important for completeness, they allow us to build prime theories:

A set Φ of (possibly open) sentences is a prime theory when

1. Φ is deductively closed.
2. If $P \vee Q \in \Phi$ then $P \in \Phi$ or $Q \in \Phi$ (or both).
3. If $\exists x.\, P \in \Phi$ then $P\{t/x\} \in \Phi$ for some term.
4. If t has sort $\mathtt{A}$ then for some $a \in \mathbb{A}$, $t = a \in \Phi$.
5. If $\mathsf{N}n.\, P \in \Phi$ then, for $P/n = P' \bullet_{y_i} (t_i)_1^k$, there is an $a \in \mathbb{A}$ such that $a\#t_i \in \Phi$ for $1 \leq i \leq k$, and $P\{a/n\} \in \Phi$.

We build a model out of Φ by taking elements to be terms and validity to be provability. Given any consistent Γ ($\Gamma \nvdash \bot$), we need to know we can consistently extend it to some Φ.

(Const$\mathbb{A}$) **and** (Small)

(Small) implies: for context Γ of language $\mathcal{L}$, if $\Gamma \vdash \exists x.\, P$ and Γ consistent, then there exists $k \in \mathbb{N}$ and $(a_i)_1^k$ such that we can extend to $\mathcal{L}'$ with constant symbol $\mathtt{f} : \mathtt{A}^k \to \mathtt{X}$ and $\Gamma, P\{f(a_i)/x\}$ is consistent.

Normally we extend with $\mathtt{c}$ and $P\{\mathtt{c}/x\}$, but in Fresh Logic $\mathtt{c}$ is equivariant.

(Const$\mathbb{A}$) implies: for every $t : \mathtt{A}$, if Γ is consistent then for some $a \in \mathbb{A}$, $\Gamma, t = a$ is consistent.

This relates to $[\![\mathtt{A}]\!] \cong \mathbb{A}$ in the semantics, which we must have for FM:

A frame α is assignements:

1. $\mathtt{X} \mapsto [\![\mathtt{X}]\!] \in \mathbf{Obj}(NOM)$ of each sort to a Nominal Set ($\mathtt{X}$ a sort, not necessarily primitive). $[\![\mathtt{X} \to \mathtt{Y}]\!] \hookrightarrow [\![\mathtt{Y}]\!]^{[\![\mathtt{X}]\!]}$ and $[\![\mathtt{A}]\!] \cong \mathbb{A}$.
2. $\mathtt{c} : \mathtt{X} \mapsto [\![\mathtt{c}]\!] \in [\![\mathtt{X}]\!]$ (c a constant symbol). $a : \mathtt{A} \mapsto a \in \mathbb{A}$.
3. $p : \mathtt{X} \mapsto [\![p]\!] \subseteq [\![\mathtt{X}]\!]$ (p a predicate constant symbol).
4. $[\![\top]\!] = \{*\}, [\![\bot]\!] = \emptyset$.
5. A valuation ϵ such that $x \in \mathbb{V} : \mathtt{X} \mapsto [\![x]\!]_\alpha \in [\![\mathtt{X}]\!]_\alpha$.

We require

1. $[\![c]\!]$ must be equivariant: $(a\ b) \cdot ([\![c]\!](us)) = [\![c]\!]((a\ b) \cdot us)$ ($\mathtt{c}$ a constant symbol).
2. $[\![p]\!]$ must be equivariant: $u \in [\![p]\!] \implies (a\ b) \cdot u \in [\![p]\!]$.

Semantics

This is an appropriate notion of model for classical Fresh Logic; for intuitionistic Fresh Logic we build a Kripke model out of these.

Non-standard is that equality is modelled by literal equality rather than a binary predicate $=$ with semantics $[\![=]\!]$. We need this, so that $[\![x]\!]$ have finite support: otherwise $\langle[\![\pi \cdot x]\!], [\![x]\!]\rangle \in [\![=]\!]$ for most π *but* $[\![\pi \cdot x]\!]=[\![x]\!]$ *only* for $\pi \not\equiv$ **Id**, endowing $[\![x]\!]$ with infinite support.

Final Slide

A sequent system, more suited to proof-search, is I believe possible, though I have not checked since I made some modifications to the logic. I have enjoyed this project a great deal.

Semantics

Define a relation $\alpha \Vdash P$ inductively on P by

1. $\alpha \Vdash p(t_1, \ldots, t_n)$ when $\langle [\![t_1]\!]_\alpha, \ldots \rangle \in [\![p]\!]_\alpha$.
2. $\alpha \Vdash t = t'$ when $[\![t]\!] = [\![t']\!]$ and $\alpha \Vdash t \# t'$ when $[\![t]\!] \# [\![t']\!]$.
3. $\alpha \Vdash P \vee Q$ when $\alpha \Vdash P$ or $\alpha \Vdash Q$. Similarly for $\wedge$.
4. $\alpha \Vdash P \Rightarrow Q$ when $\forall \beta \geq \alpha.\, \beta \Vdash P \Rightarrow \beta \Vdash Q$.
5. $\alpha \Vdash \forall x : \mathtt{X}.\, P$ when $\forall \beta \geq \alpha.\, \beta \Vdash P$.
6. $\alpha \Vdash \exists x : \mathtt{X}.\, P$ when $\exists \beta \geq \alpha.\, \beta \Vdash P$.
7. $\alpha \Vdash \text{И} n.\, P$ when $\alpha \Vdash P\{a/n\}$ for cofinitely many a.