Restart: a natural rule

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What this talk is about

I will talk about $(Restart)$, a Natural Deduction rule which moves from Intuitionistic Logic to Classical Logic, without disturbing proof-normalisation.

It turns out to have strong connections to programming.

Example deductions

$$\frac{A \quad B}{A \wedge B}\,(\wedge I) \qquad \qquad \frac{\dfrac{\forall x.\ P(x)}{P(x)}\,(\forall E) \qquad \dfrac{\forall x.\ Q(x)}{Q(x)}\,(\forall E)}{\dfrac{P(x) \wedge Q(x)}{\forall x.\ (P(x) \wedge Q(x))}\,(\forall I)}\,(\wedge I)$$

These deductions are nice, because the logical context is implicit (A and B in the first deduction, $\forall x.\ P(x)$ and $\forall x.\ Q(x)$ in the second deduction).

Curry-Howard Correspondence

The essential case for implication corresponds to β-reduction in the typed λ-calculus:

$$\dfrac{\dfrac{\begin{array}{c}[A]^1\\ \vdots\ \Psi\\ B\end{array}}{A \to B}\,(\to I)^1 \qquad \begin{array}{c}\vdots\ \Pi\\ A\end{array}}{B}\,(\to E) \qquad \Longrightarrow \qquad \begin{array}{c}\vdots\ \Pi\\ A\\ \vdots\\ B\end{array}$$

$$(\lambda x{:}A.\Psi)\Pi \qquad \longrightarrow \qquad \Psi[\Pi/x]$$

Problems with classical logic

We can extend Intuitionistic Natural Deduction to Classical Natural Deduction in various ways:

$$\frac{}{A \vee \neg A}\,(EM) \quad \frac{\neg\neg A}{A}\,(DNE) \quad \frac{}{((A \to B) \to A) \to A}\,(Pierce)$$

Problem is, these extensions are not structural (neither clearly-defined intro- nor elim-rules for anything in particular). So we lose proof-normalisation, and thereby also the clear computational content of Natural Deduction via the Curry-Howard correspondence.

Restart

$$(Restart) \qquad \frac{A}{B}$$

This is not a misprint. From A we may proceed to B.

Go on. Admit it. You're impressed.

Oh, the side-condition?

Below every occurrence of restart from A to B, there is (at least) one occurrence of A. So $\dfrac{\top}{B}$ is not valid but this is:

$$\dfrac{\dfrac{\dfrac{[A]}{B}\,(Restart)^{*}}{A \to B}\,(\to I) \qquad [(A \to B) \to A]}{\dfrac{A^{\dagger}}{((A \to B) \to A) \to A}\,(\to I)}\,(\to E)$$

The restart at * is justified at †. This is Peirce's Law.

What's going on

Natural deduction does have state, given by the undischarged assumptions.

Restart insists we return to A, but nothing stops us discharging things before we do. This turns out to give precisely the extra deductive power of classical logic:

Theorem: Intuitionistic Logic plus $(Restart)$ has the same entailment relation as Classical Logic.

Proof: One direction is the derivation above.

Proof-normalisation

Treat restart $\dfrac{A}{B}$ as an intro-rule. So it introduces whatever the top-level connective of B is.

The essential case is easy (we consider just $\longrightarrow$):

$$\dfrac{\dfrac{A}{B \longrightarrow C}\,(Restart) \qquad B}{C}\,(\longrightarrow E) \quad \Longrightarrow \quad \dfrac{A}{C}\,(Restart)$$

Call this gobbling context.

Proof-normalisation

Other essential cases are as before:

$$\dfrac{\dfrac{\begin{matrix}[A]^1\\ \vdots\ \Psi \\ B\end{matrix}}{A\to B}\,(\to I)^1 \qquad \begin{matrix}\vdots\ \Pi\\ A\end{matrix}}{B}\,(\to E) \qquad\Longrightarrow\qquad \begin{matrix}\vdots\ \Pi\\ A\\ \vdots\\ B\end{matrix}$$

However, if $A\to B$ justifies one or more instances of $(Restart)$ in the deduction above, restructure them as follows:

$$\dfrac{A\to B}{C}\,(Restart) \qquad\Longrightarrow\qquad \dfrac{\dfrac{A\to B \qquad \begin{matrix}\vdots\ \Pi\\ A\end{matrix}}{B}\,(\to E)}{C}\,(Restart)$$

The schema

$$\cfrac{\cfrac{A}{B}\,(Restart)}{\begin{array}{c}\vdots\\ \cfrac{A}{C}\,(?E)\end{array}} \qquad \Longrightarrow \qquad \begin{array}{c}\cfrac{\cfrac{A}{C}\,(?E)}{B}\,(Restart)\\ \vdots\\ \cfrac{A}{C}\,(?E)\end{array}$$

I like to call this teleportation, because any other assumptions in $(?E)$ get teleported into the derivation.

In fact, this gives derivations a notion of exception. A restart gobbles context, trying to rise to the top, and the handler of the exception passes arguments to the main thread of execution (teleportation).

The λ-calculus associated to this is very similar to Parigot's $\lambda\mu$-calculus, if you know it.

Relation to classical logic

Actually, it is obvious that restart is related to classical logic. Outstanding restarts which have yet to be justified, correspond to a notion of cocontext (the Δ in $\Gamma \vdash \Delta$ in the standard Gentzen-style presentation of classical logic).

Internalising restart: $\#$

Introduce a modality (unary connective) $\#$ with the following rules:

$$\frac{A}{\#A}\ (\#I) \qquad \frac{\#B \quad A}{B}\ (\#E) \qquad \frac{\#A_1 \quad \ldots \quad \#A_n \qquad \begin{array}{c}[A_1]^i \ldots [A_n]^i \\ \vdots \\ B\end{array}}{\#B}\ (\#P)^i$$

(See next slide for side-conditions.)

The intuition of $\#A$ is: In the next world, A.

Internalising restart: $\#$

Say $(\#E)$ is completed when A is deduced again below it. Until A is deduced again the application of $(\#E)$ is incomplete.

The side condition on $(\#E)$ is:

> (i) A must be deduced again after the application of the rule and
> (ii) no (occurrence of an) assumption on which $\#B$ depends may be discharged until A is re-deduced and
> (iii) no $(\#E)$ rule on which $\#B$ depends is completed until A is re-deduced.

Internalising restart: $\#$

So the rule is this

$$\dfrac{\#B \quad A}{\begin{array}{c} B \\ \vdots \\ A \end{array}} \; (\#E)$$

where nothing on which $\#B$ depends may be discharged until the $(\#E)$ rule is completed and no $(\#E)$ rule that is incomplete at $\#B$ is completed until A re-occurs.

These side-conditions are obtained systematically by thinking of the B below the line as happening in a procedure at a later world (which we know exists, because we assumed $\#B$). So whatever we do, we must stay later than that world until we have completed our ‘procedure’.

The meaning of $\#$

$\#A$ means $A \vee (A \rightarrow B)$ for all B.

$(Restart)$ is equivalent to assuming $\#\perp$ (i.e. there is no next world).

$\#$ inherits the same kind of proof-normalisation as restart.

Interesting questions

What programming languages do we obtain by adding restart or $\#$ to logics other than propositional logic?

What happens if we take a model of computation based on labelled transitions with state, and add a form of restart by allowing arbitrary transitions but with a global requirement to return to the originating node (but possibly with a different state). Is this a ‘theory of classical automata’?

Rules for quantifiers

$$\frac{A}{\forall x A}\ (\forall I)$$

Provided x does not occur free in any assumptions on which A depends nor (free) in any formula that is required to handle any restarts that are incomplete at A.

$$\frac{\exists x A \qquad \begin{array}{c}[A^x_c]^i \\ \vdots \\ C\end{array}}{C}\ (\exists E)^i$$

Where c is a constant which does not occur in any assumptions on which C depends, except A^x_c, nor in $\exists x A$, nor in any formula that is required to handle any restarts that are incomplete at C.

$$\frac{A}{B}$$
$$\vdots$$
$$A$$
$$\vdots$$
$$\bot$$

That is, infer B from A provided we infer A later — and then $\bot$.

(Concerning side-conditions on the universal quantifier, if x is free in A, then we cannot introduce $\forall x D$ by $\forall I$ until after A is deduced again (but we can do so before $\bot$ is deduced again as x is not free in $\bot$).)

Endpoint restart is equivalent to:

$$\forall x \neg\neg A \rightarrow \neg\neg \forall x A$$

Endpoint restart

However, note that it is purely propositional.

Intuitively, endpoint restart insists that an endpoint (final world) is visible from every world.

It adds nothing to propositional logic, because it is complete for finite Kripke models.

Overview

Restart rules are a way of strengthening logics in a ‘lite’ way. No need to move to sequents. No need, necessarily, to even mention the connectives used in an equivalent axiom-scheme.

Beautifully, the proof-normalisations are obtainable directly by treating the new rules as intro-rules.

Because of the ‘lite’ nature of restart, we wonder if it might be applied elsewhere. Exceptions of various forms are often used to implement structure-traversing connections.