A Concrete Model of Linearity and Separation

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Multiplicative conjunction and implication

$$\frac{P, Q, \Gamma \vdash \Delta}{P \otimes Q, \Gamma \vdash \Delta}\,(\otimes L) \qquad \frac{\Gamma \vdash \Delta, P \quad \Gamma' \vdash \Delta', Q}{\Gamma, \Gamma' \vdash \Delta, \Delta', P \otimes Q}\,(\otimes R)$$

$$\frac{Q, \Gamma \vdash \Delta \quad \Gamma' \vdash \Delta', P}{P \multimap Q, \Gamma, \Gamma' \vdash \Delta, \Delta'}\,(\multimap L) \qquad \frac{P, \Gamma \vdash \Delta, Q}{\Gamma \vdash \Delta, P \multimap Q}\,(\multimap R)$$

Why is this interesting?

These connectives model the idea of:

Different things happening in different parts of the universe, without interference.

To assert $P \otimes Q$ is to assert that the universe splits into two separate parts, one satisfying P and the other satisfying Q.

To assert $P \multimap Q$ is to assert that if this universe is placed separately in parallel with a universe satisfying P, then the universe as a whole satisfies Q.

(Jamie draws a picture.)

Additive conjunction and implication

$$\frac{P, Q, \Gamma \vdash \Delta}{P \wedge Q, \Gamma \vdash \Delta}\,(\wedge L) \qquad \frac{\Gamma \vdash \Delta, P \quad \Gamma \vdash \Delta, Q}{\Gamma \vdash \Delta, P \wedge Q}\,(\wedge R)$$

$$\frac{Q, \Gamma \vdash \Delta \quad \Gamma \vdash \Delta, P}{P \supset Q, \Gamma \vdash \Delta}\,(\supset L) \qquad \frac{P, \Gamma \vdash \Delta, Q}{\Gamma \vdash \Delta, P \supset Q}\,(\supset R)$$

Why is this interesting?

Well, obviously we still want to say

P and Q

and

if P then Q.

Structural rules

Note that the multiplicative and additive formulations become equivalent if we admit structural rules weakening and contraction:

$$\frac{\Gamma \vdash \Delta}{\Gamma, P \vdash \Delta}\,(\mathit{Weaken}) \qquad \frac{\Gamma, P, P \vdash \Delta}{\Gamma, P \vdash \Delta}\,(\mathit{Contract})$$

Linear logic (LL) . . .

. . . has a bang $!P$.

Banged formulae can be freely weakened and contracted. Then additive implication may be obtained by using banged propositions.

Bunched implications (BI) . . .

. . . 'orthogonally' mixes the multiplicative and additive parts.

Syntax: Colourful logic

Formulae defined by grammar:

$$P, Q ::= P \Rightarrow P \;\mid\; P \otimes P \;\mid\; \text{И}[c].\,P \;\mid\; (P - c) \;\mid\; (P + c) \;\mid\; \Box_c P \;\mid\; \bot \;\mid\; p, q, r.$$

Identify up to binding by $\text{И}[c]$. Call p, q, r propositional constants.

Colourful sugar

- $P \otimes Q$ is BI, LL mult. conj.
- $P \supset Q = И[c].\ (P{+}c \Rightarrow \Box_c Q)$ (BI add. imp).
- $\neg P = P \supset \bot$.
- $P \wedge Q = \neg(P \supset \neg Q)$ (BI add. conj).
- $\top = \bot \supset \bot$.
- $0 = И[c].\ (\top + c)$ (BI, LL mult. unit).
- $P \multimap Q = И[c].\ (P{+}c \Rightarrow Q{+}c) - c$ (BI, LL mult. imp).
- Fix some c, write $!P$ for $P - c$ (LL bang).
- $!P \multimap Q$ is LL add. imp.

Interpretation

I claim these give are (natural) translations of bunched implications and linear logic into this syntax, such that the induced semantics is sound. . .

Semantics: Multicoloured multisets

Fix a countably infinite set $a, b, c \in C$ of colours.

Write $d \in \mathcal{D}$ for the set of finite sets of colours.

For a multiset U and a function $C_U : U \to \mathcal{D}$ say C_U is a colouring of U when $\bigcup_{u \in U} C_U(u)$ is finite.

So a colouring colours elements $u \in U$ from some 'finite palette'.

Multicoloured multisets

A multicoloured multiset or universe U is a pair $(|U|, C_U : |U| \rightarrow \mathcal{D})$ of an underlying multiset $|U|$ and a colouring of $|U|$.

We may write just U for $|U|$. Write $\mathcal{U}$ for the set of universes.

Multicoloured multisets

Each colour c partitions $|U|$ into two regions:

$$\{u \in |U| \mid c \in C_U(u)\} \qquad \{u \in |U| \mid c \notin C_U(u)\}.$$

Call u uncoloured when $C_U(u) = \emptyset$.

Call U uncoloured when all $u \in U$ are uncoloured.

Interesting operations on multicoloured multisets

$U - c = (|U|, \lambda u.(C_U(u) \setminus \{c\}))$. Bleach c.

$U + c = (|U|, \lambda u.(C_U(u) \cup \{c\}))$. Paint c.

$$U \uplus U' = \left(|U| \uplus |U'|, \lambda x. \begin{cases} C_U x & x \in |U| \\ C_{U'} x & x \in |U'| \end{cases} \right).$$

Disjoint sum.

$|U| \uplus |U'|$ is multiset union.

Interesting operations on multicoloured multisets

$(a\ b)U = (|U|, \lambda u.(a\ b)C_U(u))$. Swapping action.

$(a\ b)$ is the swapping function on colours mapping a to b and vice versa, and mapping $c \neq a, b$ to itself. Its action extends pointwise to sets of colours.

$U \subseteq U'$ when $|U| \subseteq |U'|$ and $C_{U'}(u) = C_U(u)$ for all $u \in |U|$.

Multicoloured sub-multiset!

Cut-and-paste

Define $\rightarrow$ as a rewrite on multisets induced by:

- $U \rightarrow U'$ when U and U' are uncoloured and $U' \subseteq U$.
- $U \rightarrow U'$ when U and U' are uncoloured and $U' = U \uplus U$.

$\rightarrow$ cuts and pastes the uncoloured parts of U.

$\rightarrow$ is not symmetric. Any uncoloured nonempty $U \rightarrow \emptyset$ but $\emptyset \not\rightarrow U$.

Predicates

Extend the swapping action to sets of universes, pointwise. Thus

$$(a\ b)\mathcal{P} = \{(a\ b)U \ \mid\ U \in \mathcal{P}\}.$$

$\Phi(a)$ a predicate: $\text{И}a.\ \Phi(a)$ means '$\Phi(a)$ holds of all but finitely many a'.

A predicate $\mathcal{P}$ is a set of universes such that

$$\text{И}a.\ \text{И}b.\ (a\ b)\mathcal{P} = \mathcal{P}$$

See [Gabbay & Pitts 99] and later literature. Intuitively, $\mathcal{P}$ may mention many atoms, but only finitely many in a 'distinguished manner'.

Predicates

$\mathcal{U}$ (set of all universes) is a predicate.

$\emptyset$ (empty set of no universes) is a predicate.

Any finite set of universes, is a predicate.

Any set of universes mentioning colours from some finite set d, is a predicate.

The set of all universes not mentioning colours from some finite set d, is a predicate.

Order the colours $a_1, a_2, a_3, \ldots$.

The set of universes not mentioning even colours, is not a predicate — it is not fixed by $(a\ b)$ for any cofinite (complement is finite) set of colours for the a, b.

Operations on predicates

$\langle\mathcal{P}\rangle$ is the least set containing $\mathcal{P}$ and closed under $\longrightarrow$. This is a predicate (easy lemma).

$$\mathcal{P} \otimes \mathcal{Q} = \{U_P \uplus U_Q \mid U_P \in \mathcal{P},\ U_Q \in \mathcal{Q}\}$$
$$\mathcal{P} \Rightarrow \mathcal{Q} = \{U \mid U \uplus \langle\mathcal{P}\rangle \subseteq \mathcal{Q}\}$$

Note we use $\langle\mathcal{P}\rangle$, allowing cut-and-paste in $\mathcal{P}$!

Here $U \uplus \mathcal{P} = \{U \uplus U_P \mid U_P \in \mathcal{P}\}$.

More operations on predicates

$$\mathcal{P} + c = \{U + c \;\mid\; U - c \in \mathcal{P}\}$$

$$\mathcal{P} - c = \{U - c \;\mid\; U + c \in \mathcal{P}\}$$

$$[c]\mathcal{P} = \{U \;\mid\; \text{И}c'.\,(c'\ c)U \in \mathcal{P}\}.$$

$\mathcal{P} - c$ is the c-coloured part of $\mathcal{P}$, bleached.

$\mathcal{P} + c$ is the c-uncoloured part of $\mathcal{P}$, painted.

$[c]\mathcal{P}$ is the c-uncoloured part of $\mathcal{P}$, with the c-coloured part replaced by a c-coloured version of a c'-coloured part (note $(c'\ c)U \in \mathcal{P}$ iff $U \in (c'\ c)\mathcal{P}$).

For example

$(\mathcal{P} \otimes \mathcal{Q}) \pm c = (\mathcal{P} \pm c) \otimes (\mathcal{Q} \pm c)$.

$(\mathcal{P} \Rightarrow \mathcal{Q}) \pm c \neq (\mathcal{P} \pm c) \Rightarrow (\mathcal{Q} \pm c)$ in general, because
$\langle \mathcal{P} \pm c \rangle \neq \langle \mathcal{P} \rangle \pm c$ in general.

$[c](\mathcal{P} + c) = \mathcal{P} \cap \{\emptyset\}$ (here $\emptyset$ is the multiset with $|\emptyset| = \emptyset$).

$\mathcal{P}|_c = (\mathcal{P} - c) + c = \{U \in \mathcal{P} \ \mid \ U = U + c\}$. Restrict $\mathcal{P}$ to c.

If $c\#\mathcal{P}$ then $[c]\mathcal{P} = \mathcal{P}$.

$c\#[c]\mathcal{P}$.

Write $a\#\mathcal{P}$ when $\text{И}a'.\ (a'\ a)\mathcal{P} = \mathcal{P}$.

Multiplicative (separating) implication

Define $\mathcal{P} \multimap \mathcal{Q} = \text{И}c.\ [c]((\mathcal{P}{+}c \Rightarrow \mathcal{Q}{+}c) - c)$.

Lemma: $U \in \mathcal{P} \multimap \mathcal{Q}$ when $U \uplus \mathcal{P} \subseteq \mathcal{Q}$.

Lemma: $(\mathcal{P} \otimes \mathcal{Q}) \multimap \mathcal{R} \;=\; \mathcal{P} \multimap (\mathcal{Q} \multimap \mathcal{R})$.

Additive (logical) implication

$$\Box_c \mathcal{P} = \{U \uplus U{-}c \;\mid\; U \in \mathcal{P} + c\} \cup$$
$$(\mathcal{U} \setminus \{U{+}c \uplus U{-}c \;\mid\; U \in \mathcal{U}\})$$

Define $\mathcal{P} \supset \mathcal{Q} = \text{И}c.\ [c](\mathcal{P}{+}c \Rightarrow \Box_c \mathcal{Q})$.

Lemma: $\mathcal{P} \supset \mathcal{Q} = (\mathcal{U} \setminus \mathcal{P}) \cup \mathcal{Q}$.

(So $\mathcal{P} \supset \mathcal{Q}$ represents 'if $\mathcal{P}$ then $\mathcal{Q}$'.)

Bang

Suppose $\mathcal{P} = \mathcal{P}|_c$. Set $!\mathcal{P} = \mathcal{P} - c$.

$\langle\mathcal{P}\rangle = \mathcal{P}$ but in general $\langle !\mathcal{P}\rangle \neq \langle\mathcal{P}\rangle$, because with c bleached cut-and-paste is now possible.

Conclusions

I have presented a model!

I have explored the details of the constructions necessary to interpret bunched implications and linear logic within it.

I claim that this interpretation is sound.

! is interpreted as a 'you may now weaken and contract' instruction, consistent with its intuitive interpretation.

The multiplicative and additive connectives are implemented using

$$\mathcal{P} + c \qquad \mathcal{P} - c \qquad \text{И}c.\,[c]\mathcal{P} \qquad \Box_c \mathcal{P},$$

as is bang. That's three modalities, and a quantifier.

Current and future work

I am writing up all the calculations I omitted in this talk.

Note that this model is classical, whereas bunched implications is intuitionistic. This is good; classical bunched implications is an interesting topic!

Proof theory for colourful logic (sequent rules for the modalities and И)?

Extend colourful logic to predicates?

Why И?

Why did we use this quantifier? Why not simply insist that there is some finite set of colours such that for all $U \in \mathcal{P}$ the colours in U are in that finite set?

Because that would make it impossible to model negation as $\mathcal{U} \setminus \mathcal{P}$!

И lets us choose fresh names even in the presence of infinite sets. We have used this in an integral way to make the whole system work.