Resourceful truth

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Logic and Semantics Club, Edinburgh University
Friday, 15 June 2007

Thanks to Bartek Klin

My methods

I do logic, lambda-calculi, and semantics (models).

I place an emphasis on simple sets-based models.

Old-fashioned stuff really.

Structural rules

A lot of exciting developments are happening in logic for computer science.

One of them is logics that dispense with weakening and contraction — so-called structural rules.

$$\frac{\Gamma \vdash \Delta}{P, \Gamma \vdash \Delta}\,(\mathbf{Weaken}) \qquad \frac{P, P, \Gamma \vdash \Delta}{P, \Gamma \vdash \Delta}\,(\mathbf{Contract})$$

Structural rules

$(P \wedge Q) \Rightarrow P$ and $(P \wedge P) \Rightarrow P$ are theorems. We may say that $\wedge$ and $\Rightarrow$ are additive or resource-insensitive.

Without weakening and contraction such theorems are lost. Call this a multiplicative or a resource-sensitive conjunction.

Write a multiplicative conjunction as $\otimes$. An interpretation of $P \otimes P$ is

> 'I have two copies of P'.

$\otimes$ gives rise to a multiplicative implication $-\!\!*$. This 'consumes' its argument. Thus $P -\!\!* P$ is a theorem but $(P \otimes P) -\!\!* P$ is not, because we have one P left over.

Another interpretation of $P \otimes Q$ is that 'the universe splits into two parts, and P holds of this part and Q holds of the other'.

Three resource-sensitive logics

General logics: Linear logic. Bunched implications.

Domain-specific logics: Spatial logic. Separation logic.

. . . and more. . .

Relevance logics (philosophical logic)

Relevance logics model an implication $P \Rightarrow Q$ where P should be germane or relevant to the truth of Q.

For example

> ‘it’s sunny today, so the government is trying to kill me’

is true if the government is trying to kill me.

In a relevance logic the implication is false because the weather does not affect the policy of the secret service assassins.

I shall discuss only recent developments in my field.

Resource-sensitive logics

Linear logic can have $\wedge$, $\vee$, $\otimes$, $-\!\!*$, and $!$ (the exponential).

Bunched implications can have $\wedge$, $\vee$, $\Rightarrow$, $\otimes$, $-\!\!*$.

Abstract model

We want a model to interpret:

- Multiplicative connectives $\wedge$, $\vee$, $\Rightarrow$.
- Additive connectives $\otimes$, $-\!\!*$.
- The exponential $!$.

This is resourceful truth.

Resourceful truth (abstractly)

An abstract resourceful topology is a poset $(\mathcal{T}, \leq)$ such that:

- $\mathcal{T}$ contains a bottom element $\perp$. That is, $\perp \leq X$ for all $X \in \mathcal{T}$.
- $X, Y \in \mathcal{T}$ have a meet $X \wedge Y$ (the greatest $\leq$-lower bound).
- $\mathcal{X} \subseteq T$ has a join $\bigvee \mathcal{X} \in \mathcal{T}$ (the least $\leq$-upper bound of all the elements in $\mathcal{X}$, which may be infinite or empty). We write $\bigvee \mathcal{T}$ as $\top$. This is a top element.
- If $\mathcal{X} \subseteq T$ then

$$(\bigvee \mathcal{X}) \wedge Y = \bigvee \{X \wedge Y \mid X \in \mathcal{X}\}.$$

Resourceful truth (abstractly)

In addition there should be a continuous commutative monoid action I, $\otimes$.

That is, there is $I \in \mathcal{T}$ and $\otimes : (\mathcal{T} \times \mathcal{T}) \to \mathcal{T}$ such that

$$X \otimes Y = Y \otimes X \qquad X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$$

$$I \otimes Y = Y$$

$$(\bigvee \mathcal{X}) \otimes Y = \bigvee \{X \otimes Y \mid X \in \mathcal{X}\}.$$

Quantales

Quantales are similar but do not assume closure under arbitrary join.

Additive, multiplicative, exponential structure

From this we can build all the structure we need:

$$X \Rightarrow Y = \bigvee \{W \mid W \wedge X \leq Y\}$$

$$X \multimap Y = \bigvee \{W \mid W \otimes X \leq Y\}$$

$$!X = \bigvee \{W \mid W \leq X,\ W \text{ uniform}\}.$$

We call W uniform when $W \leq I$ and $W \leq W \otimes W$.

Additive, multiplicative, exponential structure

These are natural constructions.

They give the right results: soundness for bunched implications and linear logic.

Concrete models

So far so good. Now we have a candidate abstract characterisation of what we want to build. How about concrete models?

Resourceful multisets

A multiset is a set with multiplicities. For example $\{x, x\} \neq \{x\}$ because the LHS has two copies of x and the RHS has only one.

Use summation notation for multisets and ordinary sets notation for ordinary sets.

So we write the multiset $\{x, x\}$ as $x + x$ or $2 \cdot x$. Similarly $(2 \cdot x) + y$ is $\{x, x, y\}$.

In general we write $\Sigma_i a_i \cdot x_i$.

Resourceful multisets

Our multisets can have infinite multiplicities, for example $\infty \cdot x$ (infinitely many copies of x). Multiplicities are drawn from the set

$$\mathbb{N}_\infty = \{0, 1, 2, \ldots, \infty\}.$$

This has addition $+$ and multiplication $*$.

$x + \infty = \infty$.

$0 * \infty = 0$ and $x * \infty = \infty$ if $x > 0$.

Combining resourceful multisets

A normal topological space $\mathcal{T}$ is a set of sets of points (a set of open sets), closed under $\perp$, $\cap$, $\bigcup$.

Concrete resourceful topologies will be sets of points and the points are multisets. So a resourceful topology is a set of sets of multisets.

Sets of multisets enjoy a vector multiplication:

$$X \times Y = \{x + y \mid x \in X,\ y \in Y\}$$

$\times$ will model $\otimes$ — kind of. More on this in a moment.

Resourceful multiset topologies

A concrete resourceful topology is set of sets of multisets $\mathcal{T}$ such that:

- $\mathcal{T}$ contains a $\subseteq$-bottom element $\bot$.
- $\mathcal{T}$ is closed under sets intersection.

  That is, if $X, Y \in \mathcal{T}$ then $X \cap Y \in \mathcal{T}$.
- $\mathcal{T}$ is closed under infinite sets union $\bigcup$.

  That is, if $\mathcal{X} \subseteq \mathcal{T}$ then $\bigcup \mathcal{X} \in \mathcal{T}$.
- Write $\top = \bigcup \mathcal{T}$.

[deep breath] . . . and . . .

Resourceful multiset topologies

- There is a function $[\text{-}]$ mapping multisets Q (not necessarily in $\mathcal{T}$) to $[Q] \in \mathcal{T}$ such that:
  - $[\bigcup_i Q_i] = \bigcup_i [Q_i]$.
  - $[[Q] \times X] = [Q \times X]$.

  Call $[\text{-}]$ a completion and write $X \otimes Y$ for $[X \times Y]$.
- There is an $\mathsf{I} \in \mathcal{T}$ such that $\mathsf{I} \otimes X = X$ for all $X \in \mathcal{T}$.

Here $X, Y \in \mathcal{T}$ and Q and Q_i are not necessarily in $\mathcal{T}$.

Resourceful multiset topologies

Do you see how that works, or would you like me to briefly run through the proofs?

Resourceful multiset topologies

OK then.

Commutativity:

$$X \otimes Y = [X \times Y] = [Y \times X] = Y \otimes X.$$

Associativity:

$$(X \otimes Y) \otimes Z = [[X \times Y] \times Z] = [X \times Y \times Z] = [X \times [Y \times Z]] = X \otimes (Y \otimes Z).$$

Continuity:

$$(\bigcup_i X_i) \otimes Y = [(\bigcup_i X_i) \times Y] = [\bigcup_i (X_i \times Y)] = \bigcup_i [X_i \times Y] = \bigcup_i (X_i \otimes Y)$$

An example

Suppose Q is a countable set of multisets. Write $\#Q$ for the cardinality of Q taking $\#Q = \infty$ if Q is infinite. Choose some countable set of multisets $\mathbb{S}$ such that $n \cdot \mathsf{count} \in \mathbb{S}$ for all $n \in \mathbb{N}_\infty$, and no other elements mention count. Define

$$Q' = (Q \cap \mathbb{S}) \setminus \{n \cdot \mathsf{count} \mid n \in \mathbb{N}_\infty\}.$$

Let $\mathcal{T}$ be all subsets $X \subseteq \mathsf{RMult}(\mathbb{S})$ such that

$$\forall n \in \mathbb{N}_\infty.(n \leq \#X' \Leftrightarrow n \cdot \mathsf{count} \in X').$$

(So $\top = \mathbb{S}$.) Define

$$[Q] = Q' \cup \{n \cdot \mathsf{count} \mid n \leq \#Q'\}.$$

An example

- $[\bigcup_i Q_i] = \bigcup_i [Q_i]$.
- $[[Q] \times X] = [Q \times X]$.

(Honest.)

I like to think of $[Q]$ as an ‘evaluation’. Here, ‘evaluation’ is just counting.

Underlying meaning

Think of each element of $\mathcal{T}$ as a value for computation, which consists of a load of stuff.

$X \times Y$ munges all the stuff together.

$[\text{-}]$ computes a new value (conversations with Jim Lipton).

$\otimes$ is decomposed as $\times$ and $[\text{-}]$.

Let's make things even simpler

Call an accessibility a function θ such that:

- $\theta(\bigcup_i Q_i) = \bigcup_i \theta(Q_i)$.
- $\theta(\theta(Q) \times X) = \theta(Q \times X)$.
- $\theta(Q) \subseteq \top$ always.

Define

$$[Q] = \bigcap\{W \in \mathcal{T} \mid \theta(Q) \subseteq W\}.$$

Suppose $[Q] \in \mathcal{T}$ always.

Then $[\text{-}]$ is a completion.

Let's make things even simpler

Oh but where will we get θ from?

Call $\vdash$ a relation on multisets an accessibility relation when:

- If $w \vdash q' + x$ and $q' \vdash q$ then $w \vdash q + x$.
- If $w \vdash q + x$ then there is some q' such that $q' \vdash q$ and $w \vdash q' + x$.

Let $\theta(Q) = \{q' \mid q' \vdash q\}$. This generates an accessibility.

All the models I know of (e.g. Larchy-Wendling and Galmiche "Quantales as completions of ordered monoids") come from an accessibility relation $\vdash$.

Kripke!

More examples

Define $q \vdash q$ when $q \in \top$. It is easy to calculate that

$$\theta(Q) = Q \cap \top.$$

This is an accessibility. It defines partiality — restricts $X \times Y$ to $\{x + y \mid x \in X \wedge y \in Y \wedge x + y \in \mathbb{S}\}$.

More examples

Take $\mathcal{T} = \mathbb{P}(\mathsf{RMult}_{0,1}(\mathbb{S}))$ the set sets of multisets over $\mathbb{S}$ with multiplicities of 0 or 1. For example $p + q$, but not $2 \cdot p + q$.

Define $\vdash$ by

$$\Sigma_i 1 \cdot q_i \vdash \Sigma_i a_i \cdot q_i.$$

For example

$$1 \cdot p + 1 \cdot q \vdash 1 \cdot p + 3 \cdot q \qquad \text{and} \qquad 1 \cdot p + 1 \cdot q \vdash 1 \cdot p + 1 \cdot q.$$

Nominal tensor model (briefly)

It has long been commented that the nominal tensor $X \otimes Y = \{(x, y) \in X \times Y \mid \mathsf{supp}(x) \cap \mathsf{supp}(y) = \emptyset\}$ is a candidate for a multiplicative product.

I think I can make that formal in this model.

Future research

More models.

No problems with disjunction.

Explore models of linear logic in more detail — in this talk I concentrated on $\otimes$, common to linear logic and bunched implications.

New logics from this semantics?

Nominal techniques, as mentioned.