Resourceful truth

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My methods

I do logic, lambda-calculi, and semantics (models).

I try to place an emphasis on using very simple, concrete, sets-based models.

Very old-fashioned stuff really.

Structural rules

Computer science is where a lot of the most exciting developments in logic are happening nowadays.

One of the more interesting relatively recent developments has been dispensing with weakening and contraction — so-called structural rules.

$$\frac{\Gamma \vdash \Delta}{P, \Gamma \vdash \Delta}(\mathbf{Weaken}) \qquad \frac{P, P, \Gamma \vdash \Delta}{P, \Gamma \vdash \Delta}(\mathbf{Contract})$$

Structural rules

$(P \wedge Q) \Rightarrow P$ and $(P \wedge P) \Rightarrow P$ are theorems. We may say that $\wedge$ and $\Rightarrow$ are additive or resource-insensitive.

If we lose weakening and contraction then these we lose such theorems. Call this a multiplicative or a resource-sensitive conjunction.

In this case we write conjunction as $\otimes$. An interpretation of $P \otimes P$ is 'I have two copies of P'.

$\otimes$ gives rise to a multiplicative implication $\multimap$. This 'consumes' its argument. Thus $P \multimap P$ is a theorem but $(P \otimes P) \multimap P$ is not, because we have one P left over.

Another interpretation of $P \otimes Q$ is that 'the universe splits into two parts, and P holds of this part and Q holds of the other'.

Three resource-sensitive logics

Linear logic.

Bunched implications.

Spatial logic.

Also: relevance logics

Philosophical logic has beeen aware of these issues since the beginnings of logic itself. Relevance logics use multiplicative implication to a notion of implication $P \Rightarrow Q$ where P should be germane to the truth of Q.

For example 'it's sunny today, so the government is trying to kill me' is true if the government is trying to kill me.

In a relevance logic this would be false, since the weather does not affect the policy of the secret service assassins. These issues have been studied purely from the point of view of logic.

I shall speak only about recent developments in my field.

Three resource-sensitive logics

Linear logic has $\wedge$, $\otimes$, $\multimap$, and $!$.

Bunched implications has $\wedge$, $\Rightarrow$, $\otimes$, $\multimap$.

Spatial logic has $\wedge$, $\Rightarrow$, $\otimes$, $\multimap$, plus some very specific stuff for expressing properties of names and processes.

Timeline

Can we build a simple sets-based model which can interpret:

- Multiplicative connectives.
- Additive connectives.
- The linear logic exponential $!$.
- . . . and perhaps even the specific stuff of spatial logic.

This is resourceful truth.

Resourceful truth (abstractly)

An abstract topology is a poset $(\mathcal{T}, \leq)$ such that:

- $\mathcal{T}$ contains a bottom element $\bot$. That is, $\bot \leq X$ for all $X \in \mathcal{T}$.
- $X, Y \in \mathcal{T}$ have a meet $X \wedge Y$ (the greatest $\leq$-lower bound).
- $\mathcal{X} \subseteq T$ has a join $\bigvee \mathcal{X} \in \mathcal{T}$ (the least $\leq$-upper bound of all the elements in $\mathcal{X}$, which may be infinite or empty). We write $\bigvee \mathcal{T}$ as $\top$. This is a top element.
- If $\mathcal{X} \subseteq T$ then

$$(\bigvee \mathcal{X}) \wedge Y = \bigvee \{X \wedge Y \mid X \in \mathcal{X}\}.$$

Resourceful truth (abstractly)

A resourceful topology is an abstract topology with a union-preserving commutative monoid action I, $\otimes$.

That is, there is $I \in \mathcal{T}$ and $\otimes : (\mathcal{T} \times \mathcal{T}) \longrightarrow \mathcal{T}$ such that

$$X \otimes Y = Y \otimes X \qquad X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$$

$$I \otimes Y = Y$$

$$(\bigvee \mathcal{X}) \otimes Y = \bigvee \{X \otimes Y \mid X \in \mathcal{X}\}.$$

Quantales

Quantales are similar but do not assume closure under arbitrary join.

Additive, multiplicative, exponential structure

From this we can build all the structure we need:

$$X \Rightarrow Y = \bigvee\{W \mid W \wedge X \leq Y\}$$

$$X \multimap Y = \bigvee\{W \mid W \otimes X \leq Y\}$$

$$!X = \bigvee\{W \mid W \leq X,\ W \text{ uniform}\}.$$

We call W uniform when $W \leq I$ and $W \leq W \otimes W$.

Additive, multiplicative, exponential structure

These are natural constructions, and they give the right results:

If we interpret the connectives of bunched implications appropriately, we get a sound model.

Likewise if we interpret the connectives of linear logic.

Concrete models

Now how about a simple, concrete, sets-based model of resourceful topologies?

Can we see $\wedge$, $\Rightarrow$, $\otimes$, $-\!\!*$, and $!$ in action on a real set?

Resourceful multisets

A multiset is a set with multiplicities. For example $\{x, x\} \neq \{x\}$ because the LHS has two copies of x and the RHS has only one.

It is convenient to use a summation notation for multisets, and an ordinary sets notation for sets.

So we write the multiset $\{x, x\}$ as $x + x$ or $2 \cdot x$.

Similarly for $2 \cdot x + y$ and $\Sigma_i a_i \cdot x_i$ and so on.

Resourceful multisets

Our multisets can have infinite multiplicities, for example $\infty \cdot x$ (infinitely many copies of x). Multiplicities are drawn from the set

$$\mathbb{N}_\infty = \{0, 1, 2, \ldots, \infty\}.$$

This has addition $+$ and multiplication $*$.

$x + \infty = \infty$.

$0 * \infty = 0$ and $x * \infty = \infty$ if $x > 0$.

We can model 'ordinary' sets by mapping X to $\Sigma_{x \in X} \infty \cdot x$. Call such multisets simple.

Combining resourceful multisets

Scalar addition and multiplication:

$$(\Sigma_i a_i \cdot z_i) + (\Sigma_i b_i \cdot z_i) = \Sigma_i (a_i + b_i) \cdot z_i$$
$$(\Sigma_i a_i \cdot z_i) * (\Sigma_i b_i \cdot z_i) = \Sigma_i (a_i * b_i) \cdot z_i$$

Vector multiplication:

$$(\Sigma_i a_i \cdot x_i) \times (\Sigma_j b_j \cdot y_j) = \Sigma_{i,j} (a_i * b_j) \cdot (x_i + y_j).$$

Scalar addition generalises $\cup$, scalar multiplication generalises $\cap$. If the multisets are simple, $+$ and $*$ become exactly sets union and intersection.

$\times$ will model $\otimes$ — kind of. More on this in a moment.

Resourceful multiset topologies

Suppose that $\mathcal{T}$ is a set of sets of multisets (a simple multiset of simple multisets!).

Suppose that $\mathcal{T}$ is closed under infinite sets intersection $\bigcap$. So if $\mathcal{X} \subseteq \mathcal{T}$ then $\{x \mid \forall X \in \mathcal{X}. x \in X\}$ is in $\mathcal{T}$.

If Q is a set of multisets write

$$[Q] = \bigcap \{X \in \mathcal{T} \mid Q \subseteq X\}.$$

If this exists and is in $\mathcal{T}$ then it is the least $X \in \mathcal{T}$ containing Q.

For example if $\mathcal{T}$ is the set of down-closed sets of predicates of a logic then $[Q]$ is the deductive closure of Q.

Write $X \otimes Y$ for $[X \times Y]$.

Resourceful multiset topologies

$\mathcal{T}$ is a resourceful multiset topology when:

- $\mathcal{T}$ is a set of sets of multisets.
- $\mathcal{T}$ is closed under infinite sets intersection $\bigcap$.
- $\mathcal{T}$ is closed under infinite sets union $\bigcup$.
- $\times$ respects $[\text{-}]$. That is, if $[Q] = [Q']$ then $[Q \times Q''] = [Q' \times Q'']$.
- There is an $I \in \mathcal{T}$ such that $I \otimes X = X$ for all $X \in \mathcal{T}$. (Recall: $I \otimes X = [I \times X]$)

Resourceful multiset topologies

This delivers all the structure of a resourceful topology.

The closest thing to this I can find in the literature is phase spaces (a class of concrete models for linear logic). Phase spaces do not insist on closure under arbitrary unions; there exists phase spaces that are not naturally resourceful multiset topologies. Phase spaces do not use multisets (just insist on any commutative monoid).

The construction uses duals for the multiplicative structure; our construction uses least upper and lower bounds for the additive structure.

Resourceful multiset topologies deliver soundness for bunched implications and linear logic.

An example

Choose two constants p and q and let

$$\mathbb{V} = \{a \cdot \mathsf{p} + b \cdot \mathsf{q} \mid a, b \in \mathbb{N}_\infty\}.$$

Let $x = a \cdot \mathsf{p} + b \cdot \mathsf{q}$, $y = c \cdot \mathsf{p} + d \cdot \mathsf{q}$ range over elements of $\mathbb{V}$.

Define entailment by

$$x \vdash y \quad \text{when} \quad a \geq c \wedge b \geq d.$$

$\mathcal{T}$ is the set of $X \subseteq \mathbb{V}$ such that

$$\forall x, y \in \mathbb{V}.(y \in X \wedge x \vdash y) \Rightarrow x \in X.$$

An example

$\perp$ is $\emptyset$ and $\top$ is $\mathbb{V}$.

$X \otimes Y$ by the $\{w \mid \exists x \in X, y \in Y. w \vdash x * y\}$.

Here

$$(a \cdot \mathsf{p} + b \cdot \mathsf{q}) * (c \cdot \mathsf{p} + d \cdot \mathsf{q}) = (a * c) \cdot \mathsf{p} + (b * d) \cdot \mathsf{q}.$$

I is $\{x \mid x \vdash 1 \cdot \mathsf{p} + 1 \cdot \mathsf{q}\}$.

An example

Uniform elements are:

- $\mathsf{Empty} = \emptyset$.
- $\mathsf{TopRightCorner} = \{\infty \cdot \mathsf{p} + \infty \cdot \mathsf{q}\}$.
- $\mathsf{RightEdge} = \{a \cdot \mathsf{p} + \infty \cdot \mathsf{q} \mid a \geq 1\}$.
- $\mathsf{TopEdge} = \{\infty \cdot \mathsf{p} + b \cdot \mathsf{q} \mid b \geq 1\}$.
- $\mathsf{RightEdge} \cup \mathsf{TopEdge}$.

That is, in each case $W \leq I$ and $W \leq W \otimes W$.

Spatial logic (future research)

This is something of an outlier.

The underlying reason for this is that (as far as I am aware) it has no abstract models. The intended model is processes in a process language. That's it.

Perhaps things have moved on in ways I am not aware of.

In any case, I have expertise in nominal techniques which gave a (simple, sets-based) model of names and binding. It is obvious to extend the model presented here with suitable parts of nominal techniques. This is future research.

More future research

We can play.

Why take a set of sets of multisets? Why not a set of multisets of sets? Just for fun.

Build a logic with two linear structures; one describing $*$, one describing $\otimes$. What extra properties does $*$ have?

Nominal techniques, as mentioned.

The basic idea is simple. I am surprised that it has not been done before.