What sequent quantifier rules tell us about nominal semantics for logic

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Introduction

Thanks to the Leeds Logic Group for giving me the opportunity to be here today. Thank you all for coming.

Let's look at quantifier sequent rules:

$$\frac{\Gamma, \phi[a{:=}r] \vdash \psi}{\Gamma, \forall a.\phi \vdash \psi}\,(\forall\mathbf{L}) \qquad \frac{\Gamma \vdash \phi \quad (a \text{ fresh for } \Gamma)}{\Gamma \vdash \forall a.\phi}\,(\forall\mathbf{R})$$

In this talk I will outline what happens when we view these rules in the light of nominal sets and lattices.

Lattices

A lattice $\mathcal{L} = (|\mathcal{L}|, \leq)$ has

- an underlying set $x, y, z \in |\mathcal{L}|$ and
- a partial order $\leq$ with finite limits and colimits (meets $x \wedge y$ and joins $x \vee y$).

Lattices model logic: ϕ is like $x \in |\mathcal{L}|$, $x \wedge y$ is like $\phi \wedge \psi$.

If the lattice is complemented, there is an operation $\neg x$ which is like $\neg\phi$.

And so on.

Nominal lattices

Fix a countably infinite set of atoms $a, b, c, \dots \in \mathbb{A}$—atoms are like variables. Let π range over finite permutations of atoms, e.g. $(b\ a)$ swaps b and a.

A nominal lattice is a lattice, with

- a permutation action $\pi \cdot x \in |\mathcal{L}|$, and
- a notion of freshness $a\#x$.

Permutation: like renaming in syntax; $\phi[b/a]$ is like $(b\ a)\cdot x$.

Freshness: like freshness in syntax. 'a fresh for ϕ' is like $a\#x$.

Example of nominal lattices

An ordinary lattice is trivially a nominal lattice, where $\pi \cdot x = x$ and $a \# x$ always.

A set of atoms is cofinite when its complement is finite. The set of finite and cofinite set of atoms $NomPow(\mathbb{A})$ is a nominal lattice, where meets are sets intersection, joins are union, permutation is pointwise (so $\pi \cdot X = \{\pi(a) \mid a \in X\}$) and $a \# X$ when either X is finite and $a \notin X$ or X is cofinite and $a \in X$.

First-order logic syntax quotiented by derivable equivalence is a nominal lattice, where permutation is pointwise on representatives (so $\pi \cdot [\phi] = [\pi \cdot \phi]$ and $\pi \cdot \phi$ acts pointwise on all names in ϕ) and $a \# [\phi]$ when there exists some $\phi \Leftrightarrow \psi$ with $a \notin fv(\phi)$.

To category theorists: a nominal lattice is a lattice in the category of nominal sets.

Nominal sets are abstract

- Category theorists: work in Schanuel Topos.
- Set theorists: work in Fraenkel-Mostowski set theory.
- Everybody else: we have things that resemble variable symbols and you can permute/rename them and reason using freshness conditions—but they exist in (nominal) sets, not syntax.

Nominal sets are a mathematical foundation. Like any such, they display more structure the more closely you look at them. I am skimming awfully lightly over that structure.

Perhaps in five years I'll give talks where I don't worry about introducing nominal sets, any more than I worry now about introducing sets and lattices.

Fresh-finite limits

Given $x, y \in |\mathcal{L}|$ the meet or limit

$$x \wedge y \quad \text{is the greatest element in} \quad \{z \mid z \leq x,\ z \leq y\}.$$

Given $x \in |\mathcal{L}|$ and $a \in \mathbb{A}$, the $a\#$limit (a-fresh limit)

$$\forall a.x \quad \text{is the greatest element in} \quad \{z \mid z \leq x,\ a\#z\}.$$

Thus:

- $x \wedge y$ is the greatest lower bound for $\{x, y\}$, if this exists.
- $\forall a.x$ is the a-fresh greatest lower bound for $\{x\}$, if this exists.

Fresh-finite limits

Compare:

$$\frac{\Gamma \vdash \phi \quad (a \text{ fresh for } \Gamma)}{\Gamma \vdash \forall a.\phi} \qquad \frac{z \leq x \quad (a\#z)}{z \leq \forall a.x}$$

Nominal lattices with finite $a\#$limits (rather than just limits) model the right-intro rule for universal quantification.

How about the left-intro rule?

$$\frac{\Gamma, \phi[a{:=}r] \vdash \psi}{\Gamma, \forall a.\phi \vdash \psi}$$

This features substitution $\phi[a{:=}r]$, so we need to model that.

Substitution is a structural property, so we model it using nominal algebra.

Nominal algebra is algebra (logic of equalities) subject to freshness side-conditions. Axiomatisation on next slide:

Nominal algebra axiomatisation of substitution

Nominal Algebra (natural extension of nominal rewriting).
Substitution is modelled as a σ-algebra: a function $\sigma : \mathcal{L} \times \mathbb{A} \times \mathbb{A} \to \mathcal{L}$—write $\sigma(x, a, u)$ as $x[a{\mapsto}u]$—validating axioms

$$
\begin{array}{lll}
(\sigma\mathbf{a}) & & a[a{\mapsto}u] = u \\
(\sigma\mathbf{id}) & & x[a{\mapsto}a] = x \\
(\sigma\#) & a\#x \Rightarrow & x[a{\mapsto}u] = x \\
(\sigma\alpha) & b\#x \Rightarrow & x[a{\mapsto}u] = ((b\ a){\cdot}x)[b{\mapsto}u] \\
(\sigma\sigma) & a\#v \Rightarrow & x[a{\mapsto}u][b{\mapsto}v] = x[b{\mapsto}v][a{\mapsto}u[b{\mapsto}v]]
\end{array}
$$

Axioms sound and complete for syntactic model where x is ϕ and $[a{\mapsto}u]$ is ‘real’ substitution $[u/a]$ (ICTAC 2006, FAC 2008).

(To model term-formers, generalise to $\sigma : \mathcal{L} \times \mathbb{A} \times \mathcal{U} \to \mathcal{L}$ where $\mathcal{U}$ is a σ-algebra over itself.)

Back to nominal lattices

Assume the σ-action is monotone:

$$x \leq y \quad \text{implies} \quad x[a \mapsto u] \leq y[a \mapsto u].$$

Note by construction that $a \# \forall a.x$ so that by $(\sigma\#)$ $\forall a.x = (\forall a.x)[a \mapsto u]$. So

$$\forall a.x \leq x[a \mapsto u] \quad \text{for all } u.$$

Thus $\forall a.x$ behaves like $\bigwedge_u x[a \mapsto u]$ (conjunction of instances).

Back to nominal lattices

Now compare:

$$\frac{\Gamma, \phi[a{:=}r] \vdash \psi}{\Gamma, \forall a.\phi \vdash \psi}\,(\forall\mathbf{L}) \qquad \frac{z \wedge (x[a \mapsto u]) \leq y}{z \wedge \forall a.x \leq y}$$

Nominal lattices with a monotone σ-action model both left- and right-intro rules for quantification.

Summary so far:

These sequent rules ...

$$\frac{\Gamma, \phi[a{:=}r] \vdash \psi}{\Gamma, \forall a.\phi \vdash \psi}\,(\forall\mathbf{L}) \qquad \frac{\Gamma \vdash \phi \quad (a \text{ fresh for } \Gamma)}{\Gamma \vdash \forall a.\phi}\,(\forall\mathbf{R})$$

... suggest this table:

Propositional logic	Lattices	Powersets
First-order logic	NomLat$\forall\sigma$	???

Above, NomLat$\forall\sigma$ is short for "Nominal lattices with fresh-finite limits and a monotone σ-action".

Please interrupt me if you have questions.

Tarski's semantic story

Usually, we move from propositional logic to first-order logic using valuations (attributed to Tarski). The high-level shape of Tarski-style semantics is this:

$$(\text{Variables} \to \text{Denotations}) \to \text{Some Lattice.}$$

There are two problems with this:

1. To model universal quantification, the lattice (on the right) must must have all limits.
   But we just want those limits that arise from calculating universal quantifiers. So the set above is larger than necessary (cf. NF consistency proof).
2. The shape above is also not amenable to filter-style arguments and topological duality constructions.

Soundness, completeness

I'll just sketch the rest of the development:

- We can give FOL semantics in NomLat$\forall\sigma$.
- The semantics is absolute: variables in syntax map to atoms in semantics. No valuations: role of valuations is played by the σ-action.
- The semantics contains only those limits that must be there ($x \wedge y$ and $\forall a.x$). It is small.

Now for completeness; we use prime filters. What is a filter in NomLat$\forall\sigma$?

What is a filter?

A subset (which need not be finitely supported) $p \subseteq \mathcal{L}$ is a filter when:

1. $\bot \notin p$.
2. If $x \in p$ and $x \leq y$ then $y \in p$.
3. If $x \in p$ and $y \in p$ then $x \land y \in p$.
4. If $\reflectbox{N}b.(b\ a)\cdot x \in p$ then $\forall a.x \in p$.

Here $\reflectbox{N}b$ is the NEW quantifier, meaning 'for fresh b'. So this fourth condition is ($\forall$**R**) in disguise, modulo infinite support of p.

I find it hard to overstate the importance of this observation. Filters are concrete and very useful. Characterising models of first-order logic using nominal filters as above seems a big step.

The amgis-action

$\mathcal{L}$ has a sigma-action, so filters have a (dual) amgis-action ($\text{\reflectbox{$\sigma$}}$-action):

$$p[u \mapsfrom a] = \{x \mid x[a \mapsto u] \in p\}$$
$$x \in p[u \mapsfrom a] \Leftrightarrow x[a \mapsto u] \in p$$

Amgis-algebras are also useful structures. They can be axiomatised algebraically, or built concretely.

Amgis-algebras

Amgis- (ɒ-)algebras are the topological dual to σ-algebras.

- Given a σ-algebra $\mathcal{L}$, its powerset $pset(\mathcal{L})$ has an ɒ-algebra structure.
- Given an ɒ-algebra $\mathcal{P}$ (e.g. $pset(\mathcal{L})$), its powerset $pset(\mathcal{P})$ (e.g. $pset(pset(\mathcal{L}))$) has a σ-algebra structure.

Given a lattice, filters have an ɒ-algebra structure, and sets of filters regain the original σ-algebra structure.

Brief sketch of sigma and amgis

Suppose $\mathcal{L}$ has a σ-action $x[a\mapsto u]$ and suppose $p \in pset(\mathcal{L})$ and $X \in pset(pset(\mathcal{L}))$. Then we write:

$$x \in p[u\leftarrow a] \Leftrightarrow x[a\mapsto u] \in p$$
$$p \in X[a\mapsto u] \Leftrightarrow \text{И}a'.p[u\leftarrow a'] \in (a'\ a)\cdot X$$

One useful little miracle: the double powerset contains a natural model of equality (to add to the models of conjunction and quantification):

$$u{=}v = \{p \mid \text{И}c.p[u\leftarrow c] = p[v\leftarrow c]\}.$$

We can use this to extend Stone representation to first-order logic using nominal lattices and filters. I am sure that other logics are possible, and I myself have considered the untyped λ-calculus and Quine's NF.

Slogan

- As powersets model propositional logic, ...
- ...so $\reflectbox{$\sigma$}$-algebra powersets model first-order logic with equality.

Propositional logic	Lattices	Powersets
First-order logic	$\mathsf{NomLat}\forall\sigma$	Powersets of $\reflectbox{$\sigma$}$-algebras

We obtained all of this just looking carefully at the quantifier sequent rules.

Not just about first-order logic

1. The NF consistency proof uses this technology. Size issues are avoided since nominal lattices 'tend to remain small' even if they contain fresh-finite limits.
   We use the amgis constructions to obtain denotational extensionality results, dualising 'obvious' syntactic equivalences.
2. We can model the untyped λ-calculus, imitating the development above but for λ instead of $\forall$. Topological duality, for the untyped λ-calculus!
   λ is decomposed into a universal quantifier $\forall$ and an adjoint to application. The rest is obtained by paying really careful attention to the filters.

Fresh-finite limits, σ-algebras, and (powersets of) $\mathfrak{V}$-algebras are powerful concrete tools for building models.

References

- Nominal lattice semantics for first-order logic:
  Nominal semantics for predicate logic with Gilles Dowek.
  Get it at `gabbay.org.uk/papers.html#nomspl`.
- Duality for first-order logic:
  Semantics out of context: nominal absolute denotations for first-order logic and computation.
  Get it at `gabbay.org.uk/papers.html#semooc`.
- Duality for lambda-calculus:
  Representation and duality of the untyped lambda-calculus in nominal lattice and topological semantics, with a proof of topological completeness with Michael Gabbay.
  Get it at `gabbay.org.uk/papers.html#repdul`.