Consistency of Quine's NF using nominal techniques

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Introduction

Thanks to the Logic & Semantics Seminar and to Thomas Forster for inviting me here today. Thank you all for coming.

Please treat these slides as a modality for one possible talk. I will adjust technical content depending on your feedback. So please do interrupt with questions and comments.

A little background

Naive set theory has one rule; naive sets comprehension:

- If ϕ is a predicate, $\{a \mid \phi(a)\}$ is a set (the a such that ϕ). "Everything is a set, and a sets comprehension is a set."

This is inconsistent. Russell's famous 1901 paradox:

$$\{a \mid a \notin a\} \in \{a \mid a \notin a\} \quad \Leftrightarrow \quad \{a \mid a \notin a\} \notin \{a \mid a \notin a\}$$

Solutions

Solutions proposed:

- Zermelo-Fraenkel set theory (ZF sets).
  Familiar to me as a PhD student as e.g. the category of sets, or Isabelle/ZF, and so on. 'Proved' consistent by the von Neumann cumulative hierarchy model; $\varnothing$, $\mathit{powerset}(\varnothing)$, ...
- Type Theory.
  Familiar as Higher-Order Logic, ML, and so on. 'Proved' consistent by taking sets and function-sets; ι, ι^ι, $(\iota^\iota)^\iota$, ι^{ι^ι}, ...
- Quine's New Foundations (NF).

Why NF is nice

- It admits a universal set: $\{a \mid \top\}$, the set of all sets, is a set.
- My favourite NF trick: model 2 as 'the set of all two-element sets' (Frege, 1884).
- Nicer than the standard, brutal, efficient, ZF model: $2 = \{\varnothing, \{\varnothing\}\}$.
- Nicer even than the Church numeral at type α: $2_\alpha = \lambda f{:}\alpha{\to}\alpha.\lambda x{:}\alpha.f(f(x))$.

The specification of NF is concise:

- If ϕ is a stratifiable predicate, then $\{a \mid \phi(a)\}$ is a set. "Everything is a set, and stratifiable sets comprehension is a set."

(NF with urelemente is known consistent. But we lose extensionality; not everything is a set.)

Stratifiability

ϕ is stratifiable when there exists an assignment of a level to its variables such that:

- If $a{=}b$ appears in ϕ then $\mathit{level}(a) = \mathit{level}(b)$.
- If $a{\in}b$ appears in ϕ then $\mathit{level}(a){+}1 = \mathit{level}(b)$.

$a \notin a$ is unstratifiable; this blocks the comprehension of Russell's paradox.

How to semantically interpret stratifiability is part of the mystery of proving NF's consistency.

(cf. Thomas Forster's notion of stratimorphism. I may propose my own interpretation later in this talk.)

This is a proof about NF, not a proof in NF

NF is a logical theory.

This has implications to you, if you are considering how to approach my paper. The tools used in my paper—inductive reasoning, filter-style models and nominal techniques—are conducted in ZFA set theory.

You don't need to be an NF expert to read it (I'm no NF expert, and I wrote it!).

NF is indeed important as a foundation, for its universal set, for its mysterious and fascinating stratifiability condition, and for much more. But, you don't need to know about this to see that NF is consistent (though it's nice if you know a little, of course).

A simplification: TST+

Let's reduce proving consistency of NF to something easier.

First, we prove consistency of TST+ instead, which is known equivalent to NF.

Assume for each $i \in \mathbb{N}$ a set of variable symbols $\mathbb{A}^i$ (synonymously: atoms) where $level(a) = i$ if $a \in \mathbb{A}^i$. TST+ syntax is

$$\begin{aligned} s &::= a \mid \{a|\phi\} \leftarrow \text{Sets} \\ \phi &::= \bot \mid \phi \wedge \phi \mid \neg\phi \mid \forall a.\phi \mid t \in s \leftarrow \text{Predicates} \end{aligned}$$

subject to a stratification condition below.

Define $level(\{a|\phi\}) = level(a)+1$. Then a predicate is stratified when for every subterm $t \in s$ in it, $level(t)+1 = level(s)$.

TST+

NF has stratifiability. TST+ has stratification.

In NF we can't form $\{a \mid a \notin a\}$ because there is no assignment of a level to a that makes $level(a)+1 = level(a)$.

In TST+ every atom has a fixed level. We can't form any $\{a^i \mid a^i \notin a^i\}$ for any $i \in \mathbb{N}$.

TST+

Axioms of TST+ include

- the axioms you'd expect, such as $\vdash \psi \Rightarrow (\phi \Rightarrow \psi)$ and $\vdash (\forall a.\phi) \Rightarrow \phi[a{:=}s]$, plus
- typical ambiguity that for every closed ϕ

$$(\textbf{TA}) \qquad \vdash \phi \Leftrightarrow \phi^{+} \leftarrow \text{Levels can be shifted}$$

where ϕ^{+} is a copy of ϕ such that every $a \in \mathbb{A}^{i}$ is replaced with a corresponding $a^{+} \in \mathbb{A}^{i+1}$.

For instance, $\exists a^1.\top \Leftrightarrow \exists a^2.\top$. (**TA**) says levels can be shifted.

TST+ $\equiv$ NF means "stratifiability $\equiv$ (stratification + TA)".

TST$\mathbb{Z}+$

We simplify further.

Consider TST$+$ indexed over $\mathbb{Z}$ instead of $\mathbb{N}$.

Levels stretch up and also down.

Call this TST$\mathbb{Z}+$ (my terminology).

This suggests a proof-method for consistency: build a model that is symmetric under translational symmetries of $\mathbb{Z}$. (**TA**) would follow by symmetry of the model.

Another simplification: normal forms

$s{\in}\{a|\phi\}$ is equivalent to $\phi[a{\mapsto}s]$. We may reduce every predicate to one where every $\in$ has the form

$$t{\in}a,$$

for a a variable symbol (an atom). Rewrite this as

$$a{\circ}t$$

to make it look like a λ-term in head normal form (HNF). Call predicates of the form $t{\in}a$ base predicates.

Let a prepoint be a set of base predicates. So

$$p = \{a{\circ}s \mid a{\circ}s \in p\} \leftarrow \text{Prepoint is a set of normal forms}$$
$$p(a) = \{s \mid a{\circ}s \in p\} \leftarrow \text{...and also a valuation}$$

So a prepoint behaves like two things:

- Like a set of predicates in HNF.
- Like a valuation mapping atoms to sets (actually: to sets of sets syntax).

Outline of the proof is becoming clearer

We propose a model $[\![\phi]\!]$ as a set of prepoints as follows:

- $[\![s\in a]\!] = \{p\in Points \mid a\circ s \in p\}$. $\leftarrow$ From duality theory.
- $[\![\bot]\!] = \varnothing$ $\leftarrow$ As expected.
- $[\![\phi \wedge \phi']\!] = [\![\phi]\!] \cap [\![\phi']\!]$ $\leftarrow$ Ditto.
- ...

How do we interpret $\{a|\phi\}$? Let's be really simple-minded and use nominal atoms-abstraction—thus syntactic abstraction maps to semantic abstraction:

- $[\![\{a|\phi\}]\!] = [a][\![\phi]\!]$

How do we interpret $\forall a.\phi$? We use the $\reflectbox{N}$-quantifier on sets:

- $[\![\forall a.\phi]\!] = \{p \mid \reflectbox{N}a'.p \in (a'\ a)\cdot[\![\phi]\!]\}$.

Universal quantifier

If X is a set of prepoints write $\reflectbox{N}a.X = \{p \mid \reflectbox{N}a'.p \in (a'\ a)\cdot X\}$.
Then recall

$$[\![\forall a.\phi]\!] = \reflectbox{N}a.[\![\phi]\!].$$

So $p \in [\![\forall a.\phi]\!]$ when for most $a' \in \mathbb{A}^{level(a)}$, $p \in (a'\ a)\cdot[\![\phi]\!]$.

(More on what 'for most' means, shortly.)

Recall p behaves like a map $a \mapsto \{s \mid a \circ s \in p\}$. So intuitively,

$$p \in [\![\forall a.\phi]\!] \quad \text{when} \quad \reflectbox{N}a'.p \in [\![\phi]\!][a \mapsto p(a')].$$

To prove consistency of TST$\mathbb{Z}+$, we need only ensure that when we quantify over most a', $p(a')$ ranges over all possibilities for $p(a')$.

And then we're done! $\forall$ is modelled using $\reflectbox{N}$, and NF is consistent.

Quick recap

NF $\rightarrow$ TST$+$ $\rightarrow$ TST$\mathbb{Z}+$. Typical ambiguity (**TA**) follows.

TST$\mathbb{Z}+$ $\rightarrow$ TST$\mathbb{Z}+$WithNormalForms. This gives us prepoints as sets of head normal forms $a{\circ}s$.

$\{a|\text{-}\}$ $\rightarrow$ $[a]$- so abstraction maps to abstraction.

$\forall a.$- $\rightarrow$ $\text{И}a.$- so $\forall$ maps to И.

The internal syntax: surprisingly hard

Sections 3 and 4 of my paper construct the syntax of TST$\mathbb{Z}$+WithNormalForms. It's called internal syntax in the paper.

We use nominal algebra to express its properties. See Figures 1 and 2 of the paper.

A critical property is: terms have a minimum level; the level of a lowest-level atom appearing free or bound in the term. The normalising rewrite $t{\in}\{a|\phi\} \to \phi[a{:=}t]$ does not reduce this minimum level.

Thus, though levels are in $\mathbb{Z}$, in certain critical lemmas, levels are still bounded below. This gives us inductive reasoning. A typical example is Lemma 4.12.

Note on difficulty: this took six to nine months to sort out, which means I spent as much time on this, as I did on the rest of the paper put together. (min level + atoms)

Two further problems

I was disingenuous earlier. It's not enough to map $\forall$ to И. We need two more things:

1. We need to decide how to interpret substitution.
   Since $\forall a.\phi \Rightarrow \phi[a{:=}s]$, we need $[\![\forall a.\phi]\!] \subseteq [\![\phi[a{:=}s]]\!]$. Now $[\![\phi[a{:=}s]]\!]$ is just a set of prepoints.
   We need some action on sets of prepoints such that $[\![\phi[a{:=}s]]\!] = [\![\phi]\!][a{\mapsto}s]$.
2. We encounter size issues; when we make И emulate $\forall$ we need every atom to 'name' some possible value for $p(a')$.
   However, $p(a')$ is itself a set of syntax, so we may run into size issues where the cardinality of possible $p(a')$ is greater than that of the atoms available to name them.

First problem resolved: substitution on sets of points

The action $[\![\phi]\!][a{\mapsto}s]$ is off-the-shelf. We use amgis-algebras, the dual to sigma-algebras. See the paper, or my "Semantics out of Context" paper. ← Or notes at end.

- Given a set of syntax S, its powerset $pow(S)$ has an amgis-action and its double powerset $pow(pow(S))$ has a sigma-action.
- (Internal) sets and predicates are syntax so have a sigma-action. $\phi[a{:=}s]$.
- Points are sets of syntax $p = \{a{\circ}s \mid a{\circ}s \in p\}$ so have an amgis-action $p[s{\leftarrow\!\!\shortmid}a]$.
- Sets of points $[\![\phi]\!] = \{p \mid p \in [\![\phi]\!]\}$ have a sigma-action $[\![\phi]\!][a{\mapsto}s]$.

Second problem resolved: size

In finitely-supported nominal techniques, the finitely-supported powerset of atoms is countable. A similar observation, scaled up, helps with size issues here.

We set $\#\mathbb{A} = \beth_\omega$. Where $\beth_\omega$ is a cardinal that is the size of a model of Zermelo set theory, so we have 'plenty' of atoms!

We use small support instead of finite support, where X is small when $\#X < \beth_\omega$.

The small-supported powerset of syntax has the same size of syntax. (Implicit use of permission sets glossed over here.)

Filter construction

The consistency proof then reduces to a filter-style construction, where we build a prepoint p that names every possible value of $p(a')$.

'Naming' means something technical here, that involves a generous quantifier Ə, where Ə$a.\Phi$ holds when $\neg$И$a.\neg\Phi$.

Summary

- We reduce consistency of NF to a filter-like set of terms in head normal form, over syntax closely related to TST+.
- Typical ambiguity is handled by the use of $\mathbb{Z}$ for indexes instead of $\mathbb{N}$.
- Sets abstraction is not modelled using sets abstraction; it is modelled using nominal atoms-abstraction instead!
- Universal quantification is not modelled using infinite intersection; it is modelled using the NEW-quantifier instead!
- We do not use finite support; we use small support instead, where small means 'has cardinality less than $\beth_\omega$'. (i.e. is a set, not a class.)
- We handle substitution using sigma- and amgis-algebras, taken off-the-shelf from nominal duality theory for FOL and the λ-calculus.

Where do we go from here? How can we leverage this further?

- What other open problems in set theory might benefit from a new way to build filters and turn syntax-with-binders into models? The NF proof relies on $\{a \mid \phi\}$ (turns into atoms-abstraction) and $\forall$ (turns into И).
- What applications exist outside set theory (e.g. type theory) that might benefit from some modification of the semantics we already have. I just can't stop thinking of $a{\circ}s$ and how much it looks like $as_1 \ldots s_n$. A stratified λ-calculus may be lurking.
- Better 'picture' needed of the semantics. Graphs with name-generation?
- What other neat things can sigma, amgis, $[a]$-, and small support do?

References

- Consistency of Quine's NF:
  Consistency of Quine's NF using nominal techniques.
  Get it at `gabbay.org.uk/papers.html#conqnf`.
- Logic in nominal sigma-powersets:
  Semantics out of context: nominal absolute denotations for first-order logic and computation.
  Get it at `gabbay.org.uk/papers.html#semooc`.

I also suggest two less detailed but more accessible documents:

- **What sequent quantifier rules tell us about nominal semantics for logic.**
  Get it at `gabbay.org.uk/talks.html`.
- **Nominal semantics for predicate logic**.
  Get it at `gabbay.org.uk/papers.html#nomspl`.

Supplementary material

PTO

Peek into the paper: internal syntax

$$\begin{aligned} x &::= \mathtt{atm}(a) \mid [a]X \\ X &::= \mathtt{and}(\mathcal{X}) \mid \mathtt{neg}(X) \mid \mathtt{all}[a]X \mid \mathtt{elt}(x, a) \end{aligned}$$

An internal set x is either

- a variable symbol $\mathtt{atm}(a)$ (think 'a')
- a comprehension $[a]X$ (think '$\{a \mid \phi\}$').

An internal predicate X is either a conjunction ($\mathcal{X}$ is a finite possibly empty set of X), a negation, a universal quantification, or has the form $x \in a$.

This is a syntax of normal forms under the rewrite rule

$$s \in \{a \mid \phi\} \to \phi[a{:=}s].$$

Define substitution ...

Peek into the paper: the sigma-action on syntax

$$
\begin{array}{lll}
(\sigma\texttt{and}) & & \texttt{and}(\mathcal{X})[a\mapsto x] = \texttt{and}(\{X[a\mapsto x] \mid X\in\mathcal{X}\}) \\
(\sigma\texttt{neg}) & & \texttt{neg}(X)[a\mapsto x] = \texttt{neg}(X[a\mapsto x]) \\
(\sigma\texttt{all}) & b\#x \Rightarrow & (\texttt{all}[b]X)[a\mapsto x] = \texttt{all}[b](X[a\mapsto x]) \\
(\sigma\texttt{eltatm}) & a\#y, x \Rightarrow & \texttt{elt}(y,a)[a\mapsto\texttt{atm}(n)] = \texttt{elt}(y[a\mapsto\texttt{atm}(n)], n) \\
(\sigma\texttt{elt}\mathbf{a}) & & \texttt{elt}(y,a)[a\mapsto[a']X] = X[a'\mapsto y[a\mapsto[a']X]] \\
(\sigma\texttt{elt}\mathbf{b}) & & \texttt{elt}(y,b)[a\mapsto x] = \texttt{elt}(y[a\mapsto x], b) \\
(\sigma[]) & c\#x \Rightarrow & ([c]X)[a\mapsto x] = [c](X[a\mapsto x]) \\
(\sigma\mathbf{a}) & & \texttt{atm}(a)[a\mapsto x] = x \\
(\sigma\mathbf{b}) & & \texttt{atm}(b)[a\mapsto x] = \texttt{atm}(b)
\end{array}
$$

Permutative convention: a, b, c over distinct atoms.

In $(\sigma\texttt{elt}\mathbf{a})$, stratification ensures definition of substitution is inductive. X may be larger than y, but a' must have lower level than a.

Intuition of stratifiability

Imagine that each time we dereference a variable, it costs us one stratifiability dollar.

My intuition for stratifiability is that ϕ comes with a number n of stratifiability dollars (equal to the difference between the highest- and lowest-level variables in a stratification of ϕ).

When we dereference a variable, we spend one dollar and n is decremented.

Eventually we run out of dollars and then we are trapped on the left-hand side, in syntax. We can no longer afford to look up variables, and we can reason inductively with variables as a base case.

One reason NF is hard

Consider some sets comprehension $\{a \mid \phi\}$.

- This has inductive structure, since ϕ is syntax.
- It has coinductive structure where $\{a \mid \phi\} \rightarrow \{b \mid \psi\}$ when $\{b \mid \psi\} \in \{a \mid \phi\}$.

So a model of NF sets can be viewed as finding a solution, write it *NF*, to this inductive/coinductive equation:

$$SyntaxFormers(NF) \longrightarrow NF \longrightarrow Powerset(NF)$$

NF is difficult because the right-hand side loops back to the left-hand side: everything is a set—including the set's behaviour!

Problem with variables: we may dereference a variable, and we have no control over what gets put into it.

See (σ**elta**).

The fundamental equation: Sets = $[\mathbb{A}]$Predicates

NF has a universal set, so we might try to construct a model such that $X = \mathit{powerset}(X)$ (so that $X \in X$).

This raises size issues. We exploit nominal atoms-abstraction instead.

Sets comprehension is just a binder:

$$\{a \mid \phi\} \quad \forall a.\phi \quad \lambda a.t \quad \int f(a)\, da$$

So our semantics solves the equation:

$$\begin{aligned} \mathrm{NF} &= [\mathbb{A}]\mathrm{SemanticsOfPredicates} \\ [\![\{a \mid \phi\}]\!] &= [a][\![\phi]\!] \end{aligned}$$

$[a]$- is atoms-abstraction (cf my PhD). No size issues: if $\mathcal{X}$ is an infinite nominal set then $[\mathbb{A}]\mathcal{X} = \{[a]x \mid a \in \mathbb{A},\ x \in \mathcal{X}\}$ has same size as $\mathcal{X}$.

Logic in nominal powersets

How to interpret logic in a nominal powerset?

Conjunction and negation correspond to sets intersection and complement, as usual. Sets membership becomes substitution:

$$[\![\{b \mid \psi\} \in \{a \mid \phi\}]\!] = [\![\phi[a{:=}\{b \mid \psi\}]]\!].$$

Sketch of sigma and amgis

Suppose $\mathcal{X}$ has a σ-action $x[a{\mapsto}u]$ and suppose $p \in \mathit{powerset}(\mathcal{X})$ and $X \in \mathit{powerset}(\mathit{powerset}(\mathcal{X}))$. Then we write:

$$x \in p[u{\leftarrow\!\shortmid}a] \Leftrightarrow x[a{\mapsto}u] \in p$$
$$p \in X[a{\mapsto}u] \Leftrightarrow p[u{\leftarrow\!\shortmid}a] \in X$$

(More on amgis in another talk; see references at end.)

$[\![\phi]\!]$ is a set of filters, and filters are (almost) sets of predicates, so $[\![\phi]\!] \in \mathit{powerset}(\mathit{powerset}(\text{Predicates}))$.

Predicates ϕ have a σ-action (substitution), thus so does $[\![\phi]\!]$.

Logic in nominal sigma-powersets

The technical jargon:

Predicates are a σ-algebra ('set with substitution'). Sets of predicates form a dual amgis-algebra. Sets of sets of predicates restore the original σ-algebra structure.

Fact: σ-powersets of $\reflectbox{$\sigma$}$-algebras naturally interpret first-order logic with equality.

An important lemma:

$$[\![\phi]\!][a{\mapsto}u] = [\![\phi[a{:=}u]]\!].$$