Consistency of Quine's NF using nominal techniques

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Introduction

Thanks to the Manchester Logic Seminar for inviting me here today. Thank you all for coming.

Naive set theory has one rule; naive sets comprehension:

- If ϕ is a predicate, $\{a \mid \phi(a)\}$ is a set (the a such that ϕ). "Everything is a set, and a sets comprehension is a set."

This is inconsistent. Russell's famous 1901 paradox:

$$\{a \mid a \notin a\} \in \{a \mid a \notin a\} \quad \Leftrightarrow \quad \{a \mid a \notin a\} \notin \{a \mid a \notin a\}$$

Solutions

Solutions proposed:

- Zermelo-Fraenkel set theory (ZF sets).
  Familiar as e.g. "the category of sets", or Isabelle/ZF, and so on.
  'Proved' consistent by the von Neumann cumulative hierarchy model; $\varnothing$, $\mathit{powerset}(\varnothing)$, $\ldots$.
- Type Theory.
  Familiar as Higher-Order Logic, ML, and so on. 'Proved' consistent by taking sets and function-sets; ι, ι^{ι}, $(\iota^{\iota})^{\iota}$, $\iota^{\iota^{\iota}}$, $\ldots$.
- Quine's New Foundations (NF).

NF is pretty

It admits a universal set: $\{a \mid \top\}$, the set of all sets, is a set.

It admits the lovely representation of the number n as 'the set of all n-element sets' (due to Frege, 1884).

- Nicer than the standard, brutal, efficient, ZF model: $2 = \{\varnothing, \{\varnothing\}\}$.
- Nicer even than the Church numeral at type α: $2_\alpha = \lambda f{:}\alpha{\to}\alpha.\lambda x{:}\alpha.f(f(x))$.

The specification of NF is almost as concise as that of naive sets; we just add the word 'stratifiable':

- If ϕ is a stratifiable predicate, then $\{a \mid \phi(a)\}$ is a set. "Everything is a set, and stratifiable sets comprehension is a set."

Stratifiability

ϕ is stratifiable when there exists an assignment of an integer level to its variables such that:

- If $a{=}b$ appears in ϕ then $\mathit{level}(a) = \mathit{level}(b)$.
- If $a{\in}b$ appears in ϕ then $\mathit{level}(a){+}1 = \mathit{level}(b)$.

$a \notin a$ is unstratifiable; this blocks the comprehension of Russell's paradox.

$\exists a, b.a \neq b \land \forall c.c \in d \Leftrightarrow (c = a \lor c = b)$ is stratifiable, which is why we can collect all 2-element sets to form

$$2 = \{d \mid \exists a, b.a \neq b \land \forall c.(c \in d \Leftrightarrow c = a \lor c = b)\}.$$

How to interpret stratifiability is part of the mystery of proving NF's consistency.

(cf. Thomas Forster's notion of stratimorphism. I will propose my own interpretation later in this talk.)

Stratifiability

$$\begin{aligned} \mathit{Russell} &= \{a \mid a \in a\} \\ 2 &= \{d \mid \exists a, b. a{\neq}b \wedge \forall c.(c \in d \Leftrightarrow (c{=}a \vee c{=}b))\} \end{aligned}$$

Stratifiability

$$\begin{aligned} \mathit{Russell} &= \cancel{\{a^1 \mid a^0 \in a^1\}} \\ 2 &= \{d^1 \mid \exists a^0, b^0 . a^0 {\neq} b^0 \wedge \forall c^0 . (c^0 \in d^1 \Leftrightarrow (c^0{=}a^0 {\vee} c^0{=}b^0))\} \end{aligned}$$

Possible confusion: NF-the-theory versus NF-the-foundation

NF is a concise logical theory—easier to specify than, say, Higher-Order Logic.

NF is also a foundational universe. I prefer to work in Zermelo-Fraenkel and Fraenkel-Mostowksi set theory, myself.

For the purposes of this talk, NF is a logical theory to prove things about, such as consistency—not a universe to prove things in.

Thomas Forster and Randall Holmes are examples of NF-istes; they live and work in NF (I learned of NF from Thomas, who taught me logic).

Inconsistency of NF would tell us interesting things about the viability of having a universal set. So a proof of consistency is relevant beyond NF.

Proof-engineering interest to NF

ZF and type theory have size issues — classes, hierarchies of universes, cardinality restrictions, and so forth. For instance the set of all sets is not a set; the category of all sets is not a category; and so on. This is a nuisance (but an impressive + useful one; ask me).

In theorem-provers (such as COQ) hierarchies of universes appear. Similarly, a nuisance.

Type theory uses polymorphism (e.g. to define $2 = \lambda f.\lambda a.f(f(a))$). Polymorphism is like cocaine: it feels great, you want more ...then life gets complicated and you end up a wreck.

Wouldn't it be nice if we knew a universe existed in which we don't need to worry about this?

About my proof

- Stratification gives a normal form for the rewrite $x \in \{a \mid \phi\} \to \phi[a \mapsto x]$.
- $\forall$ modelled using Gabbay-Pitts 'NEW' quantifier $\reflectbox{N}$.
- Sets extensionality is handled by saturating extensionality equalities to a greatest fixedpoint.
- $\forall$-elimination ($\forall E$) $(\forall a.\phi) \Rightarrow \phi[a \mapsto x]$ is modelled by logical dual to $\reflectbox{N}$ called the 'Generous' quantifier $\reflectbox{G}$.
  Generosity also corresponds to proof-theoretic strength.
- Semantics of predicates as sets of points, in the sense of ultrafilters and Stone representation.
- Substitution is modelled nominally using σ-algebras.
- Sets are (nominal) atoms-abstractions of predicates: $\{a \mid \phi\}$ is like $\lambda a.\phi$.

About my proof (short version)

$$
\begin{aligned}
\text{Stratification} &= x \in \{a \mid \phi\} \rightarrow \phi[a \mapsto x] \text{ terminates} \\
\forall &= \text{И} \\
\text{Extensionality} &= \text{gfp of ``if } x,\ y \text{ have same elements then add } x = y\text{''} \\
(\forall \mathbf{E}) &= (\#\mathbb{A} = \#\text{Sets}) \\
[\![\phi]\!] &= \{p \in \mathit{points} \mid \phi \in p\} \\
[\![\phi[a \mapsto u]]\!] &= \{p \mid p[u \leftarrow\!\shortmid a] \in [\![\phi]\!]\} \\
\text{Sets} &= [\mathbb{A}]\text{Predicates}
\end{aligned}
$$

$\mathbb{A}$ is the set of atoms.

Syntax and the normal form

$$
\begin{array}{lcl}
s, x & ::= & \mathtt{atm}(a) \mid [a]X \\
X & ::= & \mathtt{and}(\mathcal{X}) \mid \mathtt{neg}(X) \mid \mathtt{all}[a]X \mid \mathtt{elt}(x, a)
\end{array}
$$

An internal set x is either

- a variable symbol $\mathtt{atm}(a)$ (think 'a')
- a comprehension $[a]X$ (think '$\{a \mid \phi\}$').

An internal predicate X is either a conjunction ($\mathcal{X}$ is a finite possibly empty set of X), a negation, a universal quantification, or has the form $x \in a$.

This is a syntax of normal forms under the rewrite rule

$$s \in \{a \mid \phi\} \rightarrow \phi[a \mapsto s].$$

Normal forms let us write this:

$$[\![s \in a]\!] = \{p \mid s \in a \in p\}.$$

(Peek: the sigma-action on syntax)

$$
\begin{array}{lll}
(\sigma\mathtt{and}) & & \mathtt{and}(\mathcal{X})[a{\mapsto}x] = \mathtt{and}(\{X[a{\mapsto}x] \mid X{\in}\mathcal{X}\}) \\
(\sigma\mathtt{neg}) & & \mathtt{neg}(X)[a{\mapsto}x] = \mathtt{neg}(X[a{\mapsto}x]) \\
(\sigma\mathtt{all}) & b\#x \Rightarrow & (\mathtt{all}[b]X)[a{\mapsto}x] = \mathtt{all}[b](X[a{\mapsto}x]) \\
(\sigma\mathtt{eltatm}) & a\#y,x \Rightarrow & \mathtt{elt}(y,a)[a{\mapsto}\mathtt{atm}(n)] = \mathtt{elt}(y[a{\mapsto}\mathtt{atm}(n)],n) \\
(\sigma\mathtt{elt}\mathbf{a}) & & \mathtt{elt}(y,a)[a{\mapsto}[a']X] = X[a'{\mapsto}y[a{\mapsto}[a']X]] \\
(\sigma\mathtt{elt}\mathbf{b}) & & \mathtt{elt}(y,b)[a{\mapsto}x] = \mathtt{elt}(y[a{\mapsto}x],b) \\
(\sigma[]) & c\#x \Rightarrow & ([c]X)[a{\mapsto}x] = [c](X[a{\mapsto}x]) \\
(\sigma\mathbf{a}) & & \mathtt{atm}(a)[a{\mapsto}x] = x \\
(\sigma\mathbf{b}) & & \mathtt{atm}(b)[a{\mapsto}x] = \mathtt{atm}(b)
\end{array}
$$

In $(\sigma\mathtt{elt}\mathbf{a})$, stratification ensures definition of substitution is inductive. X may be larger than y, but a' must have lower level than a.

The fundamental equation: Sets $= [\mathbb{A}]$Predicates

NF has a universal set, so we might try to construct a model such that $X = \mathit{powerset}(X)$ (so that $X \in X$).

This raises size issues. We exploit nominal atoms-abstraction instead.

Sets comprehension is just a binder:

$$\{a \mid \phi\} \quad \forall a.\phi \quad \lambda a.t \quad \int f(a)\,da \quad \ldots \text{all} \geq [a]t$$

So our semantics does this (very simple, actually):

$$\begin{aligned} \text{NF} &= [\mathbb{A}]\text{SemanticsOfPredicates} \\ [\![\{a \mid \phi\}]\!] &= [a][\![\phi]\!] \end{aligned}$$

$[a]$- is atoms-abstraction; $[\mathbb{A}]\mathcal{X} = \{[a]x \mid a{\in}\mathbb{A}, x{\in}\mathcal{X}\}$.

No size issues: if $\mathcal{X}$ is infinite then $\#[\mathbb{A}]\mathcal{X}{=}\#\mathcal{X}$.

SemanticsOfPredicates

What does SemanticsOfPredicates look like?

Digression: a filter is a set of predicates that is consistent and deductively closed. Recall standard completeness proofs by giving first-order logic (FOL) predicates semantics as sets of filters.

We do the same: $[\![\phi]\!] \in$ SemanticsOfPredicates is a set of filters, for a suitable notion of filter (which resembles a valuation environment, for a suitable non-evident notion of valuation).

So:

- $[\![\phi]\!] \in$ SemanticsOfPredicates is a set of filters.
- $[\![\{a \mid \phi\}]\!] = [a][\![\phi]\!] \in$ SemanticsOfSets.

Logic in nominal powersets

How to interpret logic in a nominal powerset?

Conjunction and negation correspond to sets intersection and complement, as usual. Sets membership becomes substitution:

$$[\![\{b \mid \psi\} \in \{a \mid \phi\}]\!] = [\![\phi[a \mapsto \{b \mid \psi\}]]\!].$$

What about quantification $[\![\forall a.\phi]\!]$?

It is known from previous work that if $\mathcal{X}$ is a nominal set with a substitution action, then so is *powerset*(*powerset*($\mathcal{X}$)) (cf. topological duality results for FOL and λ-calculus).

Sketch of sigma and amgis

Suppose $\mathcal{X}$ has a σ-action $x[a{\mapsto}u]$ and suppose $p \in \mathit{powerset}(\mathcal{X})$ and $X \in \mathit{powerset}(\mathit{powerset}(\mathcal{X}))$. Then we write:

$$x \in p[u{\leftarrow\!\!\dashv}a] \Leftrightarrow x[a{\mapsto}u] \in p$$
$$p \in X[a{\mapsto}u] \Leftrightarrow p[u{\leftarrow\!\!\dashv}a] \in X$$

More on amgis on demand: amgis is the functional preimage of underlying substitution.

$[\![\phi]\!]$ is a set of filters, and filters are (almost) sets of predicates, so $[\![\phi]\!] \in \mathit{powerset}(\mathit{powerset}(\text{Predicates}))$.

Predicates ϕ have a σ-action (substitution), thus so does $[\![\phi]\!]$.

Logic in nominal powersets

Thus general nominal abstract nonsense allows us to write the following:

$$[\![\forall a.\phi]\!] = \bigcap_u [\![\phi]\!][a\mapsto u]$$

An important lemma is that

$$[\![\phi]\!][a\mapsto u] = [\![\phi[a\mapsto u]]\!].$$

Furthermore, in the presence of generous naming of internal sets, this simplifies to:

$$[\![\forall a.\phi]\!] = \{p \mid \text{И} b.(b\ a)\cdot\phi \in p\}.$$

New and Generous

A filter p generously names x when $\reflectbox{N}a.(a{=}x \in p)$, meaning that $a{=}x \in p$ for as many atoms a as there are internal sets.

This guarantees that if $\text{И}a.\phi(a) \in p$ then $\phi(a) \in p$ for some a such that $a{=}x \in p$

The mechanics of the proof require the set of atoms to have cardinality an inacessible cardinal. In symbols: $\#\mathbb{A} = \beth_\omega$.

We unpack this further:

- $\text{И}a.\Phi(a)$ holds when $\#\{a \mid \neg\Phi(a)\} < \beth_\omega$.
- $\reflectbox{N}a.\Phi(a)$ holds when $\#\{a \mid \Phi(a)\} = \beth_\omega$.
- И and $\reflectbox{N}$ are dual: $\text{И}a.\Phi(a) \Leftrightarrow \neg\reflectbox{N}a.\neg\Phi(a)$.
- Critical property for ($\forall$**E**):
  If $\text{И}a.\Phi(a)$ and $\reflectbox{N}a.\Psi(a)$ then $\exists a.(\Phi(a) \wedge \Psi(a))$.

Summary

Predicates are a σ-algebra ('set with substitution'). Sets of predicates form a dual amgis-algebra. Sets of sets of predicates restore the original σ-algebra structure.

Fact: σ-powersets of $\rotatebox[origin=c]{180}{σ}$-algebras naturally interpret first-order logic with equality.

Furthermore, using nominal lattices and fresh-finite limits we can identify a subset of the full powerset that is necessary for modelling logic—avoids size issues.

$\forall$ is interpreted as И. Interpretation is sound if filter generously names atoms.

Summary

- Sets comprehension is modelled in both syntax and semantics by atoms-abstraction.
- Semantics based on ultrafilters in $\mathit{powerset}(\mathit{powerset}(\mathit{Syntax}))$.
- This set is large, but using nominal lattice theory we identify a small 'logical' subclass of it.
- Nominal topological duality gives logical structure off-the-shelf: conjunction, negation, substitution, and quantification (based on $\reflectbox{N}$).
- Stratifiability gives normal forms.
- Generous naming of internal sets $\reflectbox{G}$ gives $(\forall\mathbf{E})$ $\forall a.\phi \Rightarrow \phi[a{\mapsto}x]$.

Retrospective:

This is not an easy proof! However, there is a method to it. It builds on previous work. It suggests future work.

The proof is subtle but systematic: inductions, duality/filter construction, and novel but off-the-shelf material on sigma-powersets. In this sense, the proof is 'elementary'.

Elementary explanation of stratifiability condition in terms of normal forms. Perhaps more could be made of this. Perhaps more could be made of generosity.

This work is not just about NF. It is embedded in, and enriches, a larger body of mathematics. There's a basket of techniques here . . . what else could it be applied to?