Comments on my claimed proof of the Consistency of Quine's NF

Murdoch J. Gabbay

24 February 2016

Introduction

Thanks to the Informatics Colloquium in King's College for inviting me.

I will discuss my claimed proof of the consistency of NF.

- http://www.gabbay.org.uk/papers/conqnf.pdf (consistency of NF).

The technology used is related to ideas from two previous papers on duality theory.

- http://www.gabbay.org.uk/papers/repdul.pdf (Stone duality for the λ-calculus).
- http://www.gabbay.org.uk/papers/semooc.pdf (nominal representation theory for first-order logic).

These slides are more a reminder to me of what I plan to say.

Please interrupt with questions and observations.

It's more fun that way.

Two background slides on NF

Naive set theory has one rule; naive sets comprehension:

- If ϕ is a predicate, $\{a \mid \phi(a)\}$ is a set (the a such that ϕ).
  "Everything is a set, and a sets comprehension is a set."

This is inconsistent. Russell's famous 1901 paradox:

$$\{a \mid a \notin a\} \in \{a \mid a \notin a\} \quad \Leftrightarrow \quad \{a \mid a \notin a\} \notin \{a \mid a \notin a\}$$

Solutions

Solutions proposed: ZF set theory, simple type theory ... and Quine's New Foundations (NF).

For me, the attraction of NF is mostly phisolosphical: it admits a universal set: $\{a \mid \top\}$, the set of all sets, is a set. Unlike e.g. ZF (sets and classes) and simple types (hierarchy of types), everything in NF is a set.

NF admits the lovely representation of the number *n* as 'the set of all *n*-element sets' (due to Frege, 1884).

- Nicer than the ZF model: $2 = \{\varnothing, \{\varnothing\}\}$.
- Nicer even than the Church numeral at type α: $2_\alpha = \lambda f{:}\alpha\to\alpha.\lambda x{:}\alpha.f(f(x))$.

I studied this problem because I wanted to know whether the notion of set is 'safe': you just need that one concept.

There may be some potential engineering applications—ask me!

Solutions

The specification of NF is almost as concise as that of naive sets; we just add the word 'stratifiable':

- If ϕ is a stratifiable predicate, then $\{a \mid \phi(a)\}$ is a set.
  "Everything is a set, and stratifiable sets comprehension is a set."

For technical reasons my proof works with a pair of related systems, TST+ and TZT+, which replace stratifiability with an explicit stratification:

NF = TST+ = TZT+

NF is equiconsistent with a system TZT+, which is equiconsistent with TST+. Technical differences are as follows

- NF has stratifiability (see next two slides or Section 9 of the paper).
- TST+ has stratification over $\mathbb{N}$.
- TZT+ has stratification over $\mathbb{Z}$.

For the purposes of this talk the three systems are equivalent:

$$\text{NF} \;=\; \text{TZT+} \;=\; \text{TST+}.$$

Technically, my paper proves consistency of TST+/TZT+.

TZT+ syntax

For each integer $i \in \mathbb{Z}$ fix an infinite set of atoms $\mathbb{A}^i$. The raw syntax of TZT+ is:

$$\begin{array}{ll} \text{Terms} & s ::= a \in \mathbb{A}^i \mid \{a | \phi\} \quad (i \in \mathbb{Z}) \\ \text{Predicates} & \phi ::= \bot \mid \phi \Rightarrow \phi' \mid \forall a.\phi \mid s = s \mid s \in s \end{array}$$

- The level of $a \in \mathbb{A}^i$ is i.
- The level of $\{a \mid \phi\}$ is $i+1$.

(See Definition 9.2 of the paper.)

There are $\beth_\omega$ many atoms—that's as many atoms as are in a model of ZF or simple types. This is as one might expect. Cf. Goedel's incompleteness theorem.

Stratification

ϕ is stratified when:

- If $a{=}b$ appears in ϕ then $level(a) = level(b)$.
- If $a{\in}b$ appears in ϕ then $level(a){+}1 = level(b)$.

$a \notin a$ is unstratified; this blocks the comprehension of Russell's paradox:

$$Russell = \{a^1 \mid a^0 \in a^1\}$$

The set representing the number 2 is stratified, and expresses 'the set of all two-element sets':

$$2 = \{d^1 \mid \exists a^0, b^0.a^0{\neq}b^0 \wedge \forall c^0.(c^0 \in d^1 \Leftrightarrow (c^0{=}a^0 \vee c^0{=}b^0))\}$$

Stratification understood via rewriting

In my paper I note that stratification is sufficient to guarantee strong normalisation of terms and predicates under the rewrite

$$s \in \{a|\phi\} \to \phi[a \mapsto s].$$

The atomic normal form of this rewrite is $s \in a$.

Apparently this is a new observation; nobody had noticed this before. (Thomas Forster has suggested a semantic interpretation of stratification called stratimorphisms.)

This is also the only place in my proof that stratification is really used. So we can write informally

Stratification = strong normalisation.

Interpretation of predicates

How to interpret predicates?

A standard interpretation, coming from duality theory, is this:

$$[\![\phi]\!] = \{p \in points \mid \phi \in p\}$$

Strong normalisation ensures ϕ has a normal form built from atomic predicates $t{\in}a$.

Thus stratification ensures that $[\![\phi]\!]$ can be inductively decided from atomic predicates $t{\in}a$—assume the base case

$$[\![t{\in}a]\!] = \{p \in points \mid t{\in}a \in p\}$$

and build compositionally using some appropriate choice of interpration for $\neg$, $\wedge$, $\forall$, and $=$.

The easy stuff

The interpretations of $\neg$ and $\wedge$ are standard:

$$\begin{aligned} [\![\neg\phi]\!] &= \mathit{points} \setminus [\![\phi]\!] = \{p{\in}\mathit{points} \mid p \notin [\![\phi]\!]\} \\ [\![\phi{\wedge}\phi']\!] &= [\![\phi]\!] \cap [\![\phi']\!] \end{aligned}$$

So far so good.

Quantification

The $\forall$-quantifier $\forall a.\phi$ is impredicative. It quantifies over all sets s, including comprehensions $s = \{b|\psi\}$ where ψ can be arbitrarily large. So we can't do this directly:

$$[\![\forall a.\phi]\!] = \bigcap \{[\![\phi[a{:=}s]]\!] \mid \text{all } s\}.$$

Proposed solution: equate $\forall$ with the Gabbay-Pitts 'NEW' quantifier И.

$$[\![\forall a.\phi]\!] = \{p \in \mathit{points} \mid \text{И}b.\big(\phi[a{:=}b] \in p\big)\}$$

We assume $\beth_\omega$ many atoms; that's

- the number of sets in a model of ZF set theory, or
- the number of functions in a model of simple types.

If Ξ is a predicate, then И$b.\Xi$ means 'for all but κ many atoms, where $\kappa \lneq \beth_\omega$'.

Quantification

Call a set of atoms small when it has size $\lnsim \beth_\omega$. Using ZF, we'd call such a set of atoms a set to distinguish from a proper class! (Call a set large when its complement is small.)

So $[\![\forall a.\phi]\!]$ is true when $[\![\phi[a{:=}b]]\!]$ is true for most atoms, where 'most' means 'all but a ZF set of atoms', or 'vastly, overwhelmingly many—though not quite all'.

Note for experts: $[\![\forall a.\phi]\!]$ can be more beautifully written as

$$[\![\forall a.\phi]\!] = \nu a.[\![\phi]\!]$$

where ν is the NEW quantifier on nominal sets:

$$\nu a.P = \{p \mid \text{И}b.(b\ a)\cdot p \in P\}$$

Or even more succinctly we can write this:

$$\forall = \nu, \quad \text{or just} \quad \forall = \text{И}$$

Technical advantages of $\forall = \reflectbox{N}$

Why do we do this?

- We can't take $\forall$ to be directly 'for all sets', even though we intend this eventually.
  It's too impredicative and destroys inductions.
- We can't take $\forall$ to be 'for all atoms', even though we intend this eventually.
  It's too fragile; atoms behave like variables, so if the meaning of one variable changes it can destroy a universal quantification.

Forall is fragile—И is robust under small perturbations.

So we set $\forall =$ И because it gives us robustness in inductive proofs.

We can make small changes to the meanings of atoms, without affecting the meanings of universal quantifications. More on this shortly.

Recap

- Stratification = strong normalisation and normal forms are built out of atomic predicates $t \in a$.
- $\forall = И$. Gives us robustness under small pertubations.

Extensionality

We want this:

$$\forall c.(c{\in}s{\Leftrightarrow}c{\in}t) \Rightarrow s = t.$$

Using a highly technical construction, we create an equality theory ext that is a greatest fixed point such that

- if $\forall c.(c{\in}s{\Leftrightarrow}c{\in}t)$ holds then we put $s = t$ into the equality theory.

This guarantees the implication above.

The construction is technical. Note it assigns values to atoms; if $s = a$ is added to the theory, then this has the value of an assignment $a{:=}s$.

We use robustness here; this does not change the truth-value of $\forall c.(c{\in}s{\Leftrightarrow}c{\in}t)$, because $\forall = \text{И}$ and И is robust.

Extensionality with generosity

By careful technical construction we arrive at an ext in which every s is referred to by $\beth_\omega$ many atoms.

Call a set generous when it has size $\beth_\omega$.

ext generously names every s; write this Ə.

For every s,

$$Ə a.(a = s) \in \text{ext}.$$

About my proof (short version)

Interesting fact: if a set of atoms is generous then it must intersect with any И-quantified set of atoms.

So in ext, $\forall = \text{И}$ is actually true. Any s that is generously named, will be named by a И-quantifier.

So we we have finally managed to accurately model $\forall$: in ext we can write

$$\forall \Leftrightarrow \text{И} \quad \text{and} \quad \exists \Leftrightarrow \text{Ə}.$$

This is no longer a definition: it is a theorem!

We are done

$$\begin{aligned}
\text{Stratification} &= x \in \{a \mid \phi\} \rightarrow \phi[a{\mapsto}x] \text{ terminates} \\
\forall &= \text{И} \\
\text{Extensionality} &= \text{gfp of ``if } s,\ t \text{ have same elements then add } s = t\text{''} \\
(\forall\mathbf{E}) &= (\#\mathbb{A}{=}\#\text{Sets}) \\
[\![\phi]\!] &= \{p \in \mathit{points} \mid \phi \in p\}
\end{aligned}$$

Summary

This is not an easy proof! However, there is a method to it. It builds on previous work. It suggests future work.

The proof is subtle but systematic: inductions, duality/filter construction, and novel but off-the-shelf material on sigma-powersets. In this sense, the proof is 'elementary'.

Elementary explanation of stratifiability condition in terms of normal forms.

Perhaps more could be made of generosity.

This work is not just about NF. There's a basket of interesting techniques here being applied to NF. This talk only scratches the surface. I feel like Alice dropping down the rabbit-hole.

If there's time, ask me about 'the logical dual of substitution is equality'.