Consistency of Quine's NF using nominal techniques

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Introduction

Thank you all for coming.

Quine's New Foundations is a set theory proposed in 1937 whose consistency is described by the Stanford Encyclopaedia of Philosophy as the oldest outstanding consistency question.

My claimed proof is online here:
`http://www.gabbay.org.uk/papers.html#conqnf`
This document is under review.

On NF

NF has (at least) two elegant features:

1. It admits a universal set: the set of all sets is a set.
2. It uses a stratifiability condition to avoid Russell's paradox, whose meaning is mysterious.

NF is equivalent to TST+ (typed set theory with typical ambiguity), which is a slight variant.

For the purposes of this very short talk, the differences are not relevant.

Syntax & axioms of NF

Syntactic classes are atoms $a, b, c \in \mathbb{A}$, terms s, and predicates ϕ:

$$\begin{array}{lll} s, t, u & ::= & a \in \mathbb{A} \mid \{a \mid \phi\} \\ \phi & ::= & \phi \land \phi \mid \neg\phi \mid \forall a.\phi \mid t \in s \mid s = t \end{array}$$

We call $\{a \mid \phi\}$ a comprehension.

Stratifiability

ϕ is stratifiable when there exists an assignment of an integer level to its atoms such that

- if we extend to all terms by setting $\mathit{level}(\{a \mid \phi\}) = \mathit{level}(a){+}1$, then
- if $s{=}t$ appears in ϕ then $\mathit{level}(s) = \mathit{level}(t)$, and
- if $t{\in}s$ appears in ϕ then $\mathit{level}(t){+}1 = \mathit{level}(s)$.

Let's try to stratify two comprehensions,

1. first, Russell's 'set', and
2. second, the set of all two-element sets:

Stratifiability

$$\begin{aligned} \mathit{Russell} &= \{a \mid a \notin a\} \\ 2 &= \{d \mid \exists a, b. a{\neq}b \wedge \forall c.(c \in d \Leftrightarrow (c{=}a \vee c{=}b))\} \end{aligned}$$

Stratifiability

$$
\begin{aligned}
\textit{Russell} &= \{a^1 \mid a^0 \notin a^1\} \\
2 &= \{d^1 \mid \exists a^0, b^0 . a^0 \neq b^0 \wedge \forall c^0 . (c^0 \in d^1 \Leftrightarrow (c^0 = a^0 \vee c^0 = b^0))\}
\end{aligned}
$$

Axioms of NF

If we write axiom (**Ext**) for extensionality and axiom-scheme (**SC**) for stratifiable comprehension

$$(\mathbf{Ext}) \quad \forall a.\forall a.(a = b \Leftrightarrow \forall c.(c \in a \Leftrightarrow c \in b))$$
$$(\mathbf{SC}) \quad \exists a.\forall b.(b \in a \Leftrightarrow \Phi) \quad (\Phi \text{ stratifiable})$$

then we can write

$$NF = (\mathbf{Ext}) + (\mathbf{SC}).$$

In words:

$$NF = \text{extensionality} + \text{stratifiable comprehension}.$$

Structure of my proof

My proof falls into roughly three parts:

1. Understand stratification.
2. Understand $\forall$ — $\forall$ is impredicative because the universal set is a set.
3. Understand extensional equality — which uses the impredicative universal quantification.

Understanding stratification

NF's stratifiability condition implies that the following rewrite is strongly normalising:

$$s \in \{a \mid \phi\} \to \phi[a{:=}s].$$

Theorem (G.): Stratifiable terms are confluent and strongly normalising: they rewrite confluently and in finite time to a unique normal form in the syntax

$$\begin{array}{lcl} s, t, u & ::= & a \in \mathbb{A} \mid \{a \mid \phi\} \\ \phi & ::= & \phi \wedge \phi \mid \neg\phi \mid \forall a.\phi \mid t \in a \mid s = t \end{array}$$

This is the only property of stratifiability that my proof needs.

Thus, within my proof we have:

Stratification = We can work with normal forms

This equality takes us up to page 27 in my paper.

Points-based model

The denotation $[\![\phi]\!]$ of a predicate ϕ is a set of points.

A point p is a 'well-behaved' maximally consistent sets of predicates (ultrafilters).

Normal forms give us one base case off-the-shelf:

$$[\![t{\in}a]\!] = \{p \in \mathit{Points} \mid (t{\in}a) \in p\}.$$

(Without normal forms, $[\![t{\in}s]\!]$ might lead to an infinite chain as in Russell's paradox.)

The well-behavedness conditions for points — filter conditions — contain subtle conditions to do with $\forall$.

I make heavy use of nominal techniques; a mathematical foundation derived from Fraenkel-Mostowski set theory that includes names/urelemente.

Understand $\forall$

In particular, $\forall$ is characterised as follows:

$$[\![\forall a.\phi]\!] = \bigcup\{X \subseteq \mathit{points} \mid X \subseteq [\![\phi]\!],\ a\#X\}$$

In words: the meaning of $\forall a.\phi$ is the greatest set of points in the meaning of ϕ for which a is fresh.

Note this definition is independent of quantification over all terms.

The nominal interpretation of $\forall$

In this 'nominal' view, $\forall$ is a fresh colimit. In lattice terminology

$$\forall a.x = \bigvee \{z \mid z \leq x,\ a\#z\}.$$

This is consistent with a similar definition for $\wedge$:

$$x \wedge y = \bigvee \{z \mid z \leq x,\ z \leq y\}.$$

This turns up often enough, in enough different papers, in enough useful situations, that I call it the nominal interpretation of $\forall$.

The nominal interpretation of $\forall$

The benefit of the nominal interpretation is that it is untroubled by impredicativity:

- We can inductively calculate $[\![\forall a.\phi]\!]$ if we know $[\![\phi]\!]$, since ϕ is smaller than $\forall a.\phi$.
- In the presence of impredicative quantification, such as appears in NF, $\phi[a{:=}t]$ may be larger than ϕ. So we cannot necessarily calculate $[\![\phi[a{:=}t]]\!]$ even if we know $[\![\phi]\!]$.

By the end of my proof, $[\![\forall a.\phi]\!] = \bigcap_t [\![\phi[a{:=}t]]\!]$ is true.

But this is a nontrivial theorem.

This takes us to page 52 of my paper. It remains to understand extensional equality.

Equality

NF has two primitive predicate forms:

- $t \in s$.
- $s = t$.

These are equally expressive:

- $s = t$ maps to $a \in \{b \mid s = t\}$ (for fresh dummy variables a and b) and
- $t \in s$ maps to $\{b \mid t \in s\} = \{b \mid \top\}$ (for a fresh dummy variable a).

Extensionality 'likes' ultrafilters based on the equality predicate-form, but the rest of the paper (dealing with $\neg$, $\wedge$, $\forall$, and $\in$) 'likes' ultrafilters based on the sets membership predicate-form.

The final third of the proof

The final third of the paper shows how to move between two notions of filter, one based on $\in$ and one based on $=$.

The proof of consistency for NF then reduces to constructing a specific $=$-style filter which, when translated to $\in$-style, is a point.

This construction relies heavily on nominal techniques once again. Small support properties are used to place strong bounds on the size of small-supported powersets.

The final third of the proof

I will give just a taste of how it works: in nominal sets, both $\mathbb{A}$ the set of names and $powerset_{fs}(\mathbb{A})$ are countable; basically this is because any finitely-supported set of atoms is either finite or cofinite.

A generalisation of this result gives us that if $\mathbb{A}$ is 'large' and X is a 'large' nominal set then, in a certain sense, $powerset_{ss}(X)$ the small-supported set of subsets of X is also 'large' (and not 'larger').

In other words, small-supported powersets have the same cardinality as the original set. This use of support to control cardinalities is crucial.

This is the most technical part of the paper, and is the only part of the construction to be quite specific to consistency of NF. If interested, trace uses of Lemma 10.27 in my paper.

Conclusions

The proof of consistency of NF breaks down into three distinct pieces:

1. A theory of normal forms based on stratifiability.
2. A theory of universal quantification $\forall$ differing from standard Tarski semantics with the advantage of 'playing nice' with impredicativity.
   See my previous Logic Colloquium talk, of Monday.
3. Concrete counting arguments on nominal powersets, noting that nominal powersets do not get large 'too quickly' because of support restrictions.

Each of these three points has, I feel, independent interest.

Supplement:

Go catch a very large set of atoms...

If X is a set write $\#X$ for the cardinality of X.

Let $\beth_0 = \#\mathbb{N}$. Write $\beth_\omega$ for the least cardinal larger than $\#\mathit{powerset}^n(\mathbb{N})$ for every $n \in \mathbb{N}$. So:

$$\begin{aligned} & \beth_0 = \#\mathbb{N} \\ \leq\ & \beth_1 = \#\mathit{powerset}(\mathbb{N}) \\ \leq\ & \beth_2 = \#\mathit{powerset}(\mathit{powerset}(\mathbb{N})) \leq \cdots \leq \beth_\omega . \end{aligned}$$

Fix a large (size $\beth_\omega$) set of atoms $\mathbb{A}$.

- Write $\text{И}a.\Phi(a)$ when Φ holds of all but $\kappa \lneq \beth_\omega$ many atoms. Read this as 'for new a, $\Phi(a)$'.
- Write $\text{Ə}a.\Phi(a)$ when Φ holds of $\kappa = \beth_\omega$ many atoms. Read this as 'for generously many a, $\Phi(a)$'.

Syntax normalisation procedure

Consider the rewrite on terms and predicates:

$$s \in \{a \mid \phi\} \rightarrow \phi[a \mapsto s].$$

Justified by the intuition that s is in the set of a such that ϕ if and only if $\phi[a \mapsto s]$.

Theorem: Stratifiable terms are confluent and strongly normalising under this rule. That is, they rewrite confluently and in finite time to a unique normal form.

Proof sketch: Confluence is routine. Termination follows by rewriting innermost highest level reducts. Use a multiset lexicographic ordering, which is well-founded.

Syntax of normal forms

We can easily characterise normal forms:

$$\begin{array}{lll} s, t, u & ::= & a \in \mathbb{A} \mid \{a \mid \phi\} \\ \phi & ::= & \phi \wedge \phi \mid \neg\phi \mid \forall a.\phi \mid t \in a \end{array}$$

Note the $t \in a$ on the far right; this is the base case of induction on normalised syntax.

Let a prepoint $p \in$ Prepoint be a set of assertions of the form $t \in a$. Then we provisionally interpret $t \in a$ by

$$[\![t \in a]\!] = \{p \in \text{Prepoint} \mid (\text{t}\in\text{a}) \in \text{p}\}.$$

Now we want to interpret $\wedge$, $\neg$, $=$, and $\forall$, in the syntax above in such a way as to validate all the axioms of NF.

This will be our model.

Overview of our model

Our model will interpret a predicate as a set of points, where a point is a prepoint plus conditions.

$$\phi \longmapsto [\![\phi]\!] \in \mathit{powerset}(\mathit{powerset}(\{t{\in}a \mid \text{all } t, a\})).$$

So how to interpret logical connectives $\wedge$, $\neg$, $=$, and $\forall$?

Much of this was addressed in [semooc] and [repdul]. I will sketch how it works.

Logic in nominal powersets (propositional part; high-level view)

Conjunction and negation correspond to sets intersection and complement, as usual.

$$[\![\phi\wedge\psi]\!] = [\![\phi]\!] \cap [\![\psi]\!] \qquad [\![\neg\phi]\!] = \mathit{Points} \setminus [\![\phi]\!]$$

(I haven't said which prepoints are points, or proved that any points exist.)

Sets membership becomes substitution, thanks to our rewrite rule:

$$[\![\{b \mid \psi\}\in\{a \mid \phi\}]\!] = [\![\phi[a\mapsto\{b \mid \psi\}]]\!].$$

(This isn't trivial to check.)

Logic in nominal powersets (quantifiers)

What about quantification $[\![\forall a.\phi]\!]$? Following [semooc,repdul] we write:

$$[\![\forall a.\phi]\!] = \{p \mid \text{И} b.(b\ a)\cdot\phi \in p\}.$$

It turns out that this has many equivalent presentations, including:

$$[\![\forall a.\phi]\!] = \bigcup\{X' \subseteq [\![\phi]\!] \mid a\#X'\}.$$

Thus $[\![\forall a.\phi]\!]$ is the greatest subset of $[\![\phi]\!]$ for which a is fresh, in the sense of nominal sets.

This characterisation of quantification uses only И and #. It does not depend on substitution!

Logic in nominal powersets (quantifiers)

This

$$[\![\forall a.\phi]\!] = \bigcup \{X' \subseteq [\![\phi]\!] \mid a\#X'\}$$

guarantees that:

$$\frac{}{[\![\forall a.\phi]\!] \subseteq [\![\phi]\!]} \qquad \frac{[\![\psi]\!] \subseteq [\![\phi]\!] \quad (a\#\psi)}{[\![\psi]\!] \subseteq [\![\forall a.\phi]\!]}$$

Note we do not use the familiar Tarski semantics that forall = 'for every possible value'. This would read as follows:

$$[\![\forall a.\phi]\!] = \bigcap_u [\![\phi[a\mapsto u]]\!].$$

That depends on substitution. We can't do that in NF because NF is impredicative and u may be a comprehension $\{a \mid \psi\}$ where ψ is larger than ϕ — taking $\phi[a\mapsto u]$ in a definition would be unhealthy for inductive quantities.

Logic in nominal powersets (quantifiers)

The nominal semantics of $\forall$ works generally, just like conjunction and complement.

If $X, Y \subseteq \mathcal{X}$ are subsets of a nominal set $\mathcal{X}$ we can define

$$\forall a.X = \bigcup \{X' \subseteq X \mid a\#X'\}$$

and then

$$\frac{}{\forall a.X \subseteq X} \qquad \frac{Y \subseteq X \quad (a\#Y)}{Y \subseteq \forall a.X}.$$

Thus $\forall a.X$ is the greatest subset of X for which a is fresh.

(This generalises further to nominal lattices; see [semooc].)

Logic in nominal powersets (quantifiers)

Given an extra consistency condition on prepoints called generous naming of internal sets we obtain a theorem (Theorem 8.15 in the paper):

$$[\![\forall a.\phi]\!] = \bigcap_u [\![\phi[a \mapsto u]]\!].$$

So by the end of my paper, $\forall$ is doing what we expect and quantifying over all terms.

It matters that this is a theorem, not a definition: the ϕ on the right is smaller than the $\forall a.\phi$ on the left in

$$[\![\forall a.\phi]\!] = \bigcup\{X' \subseteq [\![\phi]\!] \mid a\#X'\}.$$

So this is suitable for an inductive definition; the first equality above is not, in NF.

Substitution

An important lemma is that

$$[\![\phi[a \mapsto u]]\!] = [\![\phi]\!][a \mapsto u].$$

This is non-trivial to prove.

Indeed, it is also non-trivial to state. What is $[\![\phi]\!][a \mapsto u]$?

We know that $[a \mapsto u]$ applied to syntax ϕ is.

What is $[a \mapsto u]$ applied to a set of (pre)points like $[\![\phi]\!]$?

Substitution

Suppose $\mathcal{X}$ has a σ-action $x[a\mapsto u]$. Suppose $p \in \mathit{powerset}(\mathcal{X})$ and $X \in \mathit{powerset}(\mathit{powerset}(\mathcal{X}))$ $[\![\phi]\!]$ is one of these X).

Then define:

$$x \in p[u\leftarrow\!\!\dashv a] \Leftrightarrow x[a\mapsto u] \in p$$
$$p \in X[a\mapsto u] \Leftrightarrow \text{И}b.(p[u\leftarrow\!\!\dashv b] \in (b\ )\cdot X)$$

Amgis $[u\leftarrow\!\!\dashv a]$ is the functional preimage of underlying substitution. The σ-action on X is obtained from the amgis action on p.

Studying the two definitions above is a talk in itself. The bottom line is:

$[\![\phi]\!][a\mapsto u]$ is obtained by 'lifting' $\phi[a\mapsto u]$ as above.

New and Generous

A filter p generously names x when $\text{Ə}a.(a{=}x \in p)$, meaning that $a{=}x \in p$ for $\beth_\omega$ many atoms a.

This guarantees that if $\text{И}a.\phi(a) \in p$ then $\phi(a) \in p$ for some a such that $a{=}x \in p$

The mechanics of the proof require the set of atoms to have cardinality $\#\mathbb{A} = \beth_\omega = \bigcup_{i<\omega} \#2^i$.

We unpack this further:

- $\text{И}a.\Phi(a)$ holds when $\#\{a \mid \neg\Phi(a)\} < \beth_\omega$.
- $\text{Ə}a.\Phi(a)$ holds when $\#\{a \mid \Phi(a)\} = \beth_\omega$.
- И and Ə are dual: $\text{И}a.\Phi(a) \Leftrightarrow \neg\text{Ə}a.\neg\Phi(a)$.

The technical bits: equality and quantification

The technical rubber hits the mathematical road around Definition 11.19, Proposition 11.30, and Definition 12.37.

We require extensionality, that

$$\forall \phi, s, t, p. \big(p \in [\![s{=}t]\!] \Rightarrow (p \in [\![\phi[a{:=}s]]\!] \Leftrightarrow p \in [\![\phi[a{:=}t]]\!])\big).$$

We enforce this by an inductive construction to build a maximally consistent extensional set of equalities $s{=}t$.

We also require generous naming of internal sets, that for every p and s,

$$\reflectbox{N}a.\forall t. \big(p \in [\![t{\in}a]\!] \Leftrightarrow p \in [\![t{\in}s]\!]\big).$$

We enforce this by another induction generating a maximally consistent set of $(t{\in}a)$s.

About my proof

- Stratifiability gives a normal form for the rewrite $x \in \{a \mid \phi\} \rightarrow \phi[a{\mapsto}x]$.
- $\forall$ modelled using nominal limits.
- Sets extensionality handled by saturating extensionality equalities to a greatest fixedpoint.
- $\forall$-elimination $(\forall E)$ $(\forall a.\phi) \Rightarrow \phi[a{\mapsto}x]$ modelled by logical dual to И called the 'Generous' quantifier Ә.
  Generosity also corresponds to proof-theoretic strength.
- Semantics of predicates as sets of points, where a point is a maximally consistent set of predicates.
- Substitution modelled using ϭ-algebras.
- Comprehension = atoms-abstractions. So $[\![\{a \mid \phi\}]\!] = [a][\![\phi]\!]$.
- Atoms extensionally equal to, but not syntactically identical to, comprehensions: $[\![a]\!] = [\![\{b \mid b{\in}a\}]\!]$.
- Two equivalent, but structurally distinct, notions of 'maximally consistent sets': one designed for = and the other for $(\forall E)$.

Cheat-sheet

$$
\begin{aligned}
\text{Stratifiability} &= x \in \{a \mid \phi\} \to \phi[a \mapsto x] \text{ terminates} \\
\forall &= \text{И} \\
\exists &= \text{Ɔ} \\
\text{Extensionality} &= \text{gfp of “if } x,\, y \text{ have same elements then add } x = y\text{”} \\
(\forall \mathbf{E}) &= (\#\mathbb{A} = \#\text{Sets}) \\
[\![\phi]\!] &= \{p \in \mathit{points} \mid \phi \in p\} \\
[\![\phi[a \mapsto u]]\!] &= \{p \mid p[u \leftarrow a] \in [\![\phi]\!]\} \\
\text{Sets} &= [\mathbb{A}]\text{Predicates}
\end{aligned}
$$

$\mathbb{A}$ is the set of atoms.