Equivariant ZFA with Choice: a position paper

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Motivation

Nominal techniques assume a set $a, b, c, \ldots \in \mathbb{A}$ of *atoms*; elements that can be compared for equality but which have few if any other properties. What is a mathematical foundation for this?

Suggestion: Zermelo-Fraenkel Set theory with Choice (ZFC)

Model atoms as $\mathbb{N} = \{0, 1, 2, \ldots\}$. For more atoms use *powerset*$(\mathbb{N})$.
Advantage: Simple.
Disadvantage: Doesn't work.
Problem is that atoms should be
infinite, distinguishable, and interchangeable.
This is called **equivariance**.
Numbers are infinite, distinguishable, but not interchangeable (equivariant).

Suggestion: Fraenkel-Mostowski set theory (FM sets)

Model atoms as a set of atoms $\mathbb{A} = \{a, b, c, \ldots\}$.
Insist on **finite support** axiom (technical).
Advantage: Beautiful.
Disadvantage: Finite support axiom too strong.
Problem is, we want non-finitely-supported elements. The following are inconsistent with FM:

- "There exists a total ordering on $\mathbb{A}$";
- "Every set can be well-ordered".

Suggestion: Zermelo-Fraenkel set theory with Atoms and Choice (ZFAC)

Model atoms as a set of atoms $\mathbb{A} = \{a, b, c, \ldots\}$. Do not insist on finite support.
Advantage: Also beautiful.
Disadvantage: Equivariance is a scheme of theorems; equivariance for a predicate ϕ costs n to prove, where n is the size of ϕ. Leads to quadratic blowup in mechanised development, and stalled development.

Suggestion: Equivariant ZFAC (EZFAC)

Model atoms as a set of atoms $\mathbb{A} = \{a, b, c, \ldots\}$. Do not insist on finite support. Add equivariance as an axiom-scheme (even though it is derivable anyway).
Advantage: Goldilocks: we get Choice, and Equivariance is cheap.
Disadvantage: What disadvantage?

We have Choice and the following are derivable in EZFAC:

- "There exists a total ordering on $\mathbb{A}$";
- "Every set can be well-ordered (even if it mentions atoms)".

FM is trivially a subuniverse of EZFAC, so we can do everything we can do in FM, at nearly zero overhead.

EZFAC axioms

(AtmEmp)	$t \in s \Rightarrow s \notin \mathbb{A}$	
(EmptySet)	$t \notin \varnothing$	
(Ext)	$s, s' \notin \mathbb{A} \Rightarrow (\forall \mathrm{b}.(\mathrm{b} \in s \Leftrightarrow \mathrm{b} \in s')) \Rightarrow s = s'$	
(Pair)	$t \in \{s, s'\} \Leftrightarrow (t = s \vee t = s')$	
(Union)	$t \in \bigcup s \Leftrightarrow \exists \mathrm{a}.(t \in \mathrm{a} \wedge \mathrm{a} \in s)$	
(Pow)	$t \in \mathit{pset}(s) \Leftrightarrow t \subseteq s$	
(Ind)	$(\forall \mathrm{a}.(\forall \mathrm{b}{\in}\mathrm{a}.\phi[\mathrm{a}{:=}\mathrm{b}]) \Rightarrow \phi) \Rightarrow \forall \mathrm{a}.\phi \; \mathit{fv}(\phi) = \{\mathrm{a}\}$	
(Inf)	$\exists \mathrm{c}.\varnothing \in \mathrm{c} \wedge \forall \mathrm{a}.\mathrm{a} \in \mathrm{c} \Rightarrow \mathrm{a} \cup \{\mathrm{a}\} \in \mathrm{c}$	
(AtmInf)	$\neg(\mathbb{A} \subseteq_{\mathit{fin}} \mathbb{A})$	
(Replace)	$\exists \mathrm{b}.\forall \mathrm{a}.\mathrm{a} \in \mathrm{b} \Leftrightarrow \exists \mathrm{a}'.\mathrm{a}' \in u \wedge \mathrm{a} = F(\mathrm{a}')$	
(Choice)	$\varnothing \neq (\mathit{pset}^*(s) \to s)$	pset^* nonempty powerset
(Equivar)	$\forall \mathrm{a}{\in}\mathit{Perm}.(\phi \Leftrightarrow \mathrm{a}{\cdot\cdot}\phi)$	

The permutation action

A **permutation** π is a bijection on atoms $\mathbb{A}$.
Define an inductive **permutation action** $\pi \cdot x$ by:

- $\pi \cdot \mathrm{a} = \pi(\mathrm{a})$ if $\mathrm{a} \in \mathbb{A}$ and
- $\pi \cdot \mathrm{a} = \{\pi \cdot \mathrm{b} \mid \mathrm{b} \in \mathrm{a}\}$ if $\mathrm{a} \notin \mathbb{A}$.

Examples, where $a, b \in \mathbb{A}$:

$\pi \cdot \{a, b\} = \{\pi(a), \pi(b)\}$ $\;\pi \cdot \mathbb{A} = \mathbb{A}$ $\;\pi \cdot (\mathbb{A} \setminus \{a, b\}) = \mathbb{A} \setminus \{\pi(a), \pi(b)\}$.

Equivariance (Equivar)

$\pi {\cdot\cdot} \phi$ is ϕ with every free variable a replaced with $\pi {\cdot\cdot} \mathrm{a}$.
Examples:

- $\pi {\cdot\cdot} (\mathrm{a} \in \mathrm{b}) = \pi \cdot \mathrm{a} \in \pi \cdot \mathrm{b}$.
- $\pi {\cdot\cdot} (\mathrm{a} \in \mathit{pset}^*(\mathrm{a})) = \pi \cdot \mathrm{a} \in \mathit{pset}^*(\pi \cdot \mathrm{a})$.
- $(\mathrm{a}\ \mathrm{b}) {\cdot\cdot} (\mathrm{a} = \mathrm{b}) = (\mathrm{b} = \mathrm{a})$, where $(\mathrm{a}\ \mathrm{b})$ is the **swapping** permutation, transposing a and b.

Biinterpretability

ZFC, ZFAC, FM, and EZFAC are *biinterpretable*: any model of one can be embedded in a model of another; anything we express in one theory can be translated easily to an assertion in another.
However, 'biinterpretable' does not mean 'the same'. Roman numerals are biinterpretable with arabic numerals; C is biinterpretable with ML; but they make things easier or harder in different ways, and powerfully affect how we think.

Conclusion

- FM is mathematically too strong,
- ZFC is too weak, and
- ZFAC does not scale (quadratic slowdown).

EZFAC may be a suitable foundation for formalising nominal arguments: as the logic underlying a theorem-prover, or as a foundation for the reader's next paper.

To Learn More, See...

Murdoch Gabbay *Equivariant ZFA and the foundations of nominal techniques*. Submitted. arXiv preprint arxiv.org/abs/1801.09443.

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