Equivariant ZFA with Choice (EZFAC)

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Thanks to the organisers.

13 April 2018

What is EZFAC?

EZFAC is a ‘mathematical foundation with symmetry’.

Imagine you have data with the following properties:

- It is infinite / very large — but only because it is built out of many identical units (‘atoms’).
  With the symmetry stripped out, it admits a finite / much smaller representation.
- You want to dynamically allocate extra units, expanding your dataset.

Examples follow.

Examples

- **Memory.** Memory is a collection of addressable units in some virtual space. We can allocate more memory.
- **Fermions.** Electrons are fermions. We have many electrons. They are interchangeable; we never say 'but that isn't the electron you gave me yesterday!'.
  New electrons may be created.
- **Names in syntax.** The term $\lambda a.\lambda b.ba$ is α-equivalent to $\lambda b.\lambda a.ab$. A term quotiented by α-equivalence is an infinite symmetry equivalence class.
  Binders may be added freely: given t we may form $\lambda a.t$.
- **Nonces. Channel names. . . .**

How do foundations impact on this?

I'll use HOL as a runnning example.

Higher-Order Logic = simply-typed λ-calculus over *bool* and *nat*.

Types are $\alpha ::= \mathit{bool} \mid \mathit{nat} \mid \alpha \to \alpha$.

Axioms make *bool* behave like truth-values and *nat* behave like numbers.

bool and *nat* are ordered sets. Functions are ordered lexicographically.

The HOL universe admits no nontrivial automorphisms. It is rigid.

Typical foundations are rigid

Such a universe struggles to represent the symmetries above.

Our options are:

- Work with equivalence classes.
- Work with representatives.
- Use domain-specific tricks to compact a symmetry class down to some other small representation.

But the problem is . . .

and this imposes costs.

. . . this is expensive and may not scale.

Choosing an asymmetric representation of a symmetric object exposes you to:

- **Error.** You accidentally do something you shouldn't with your representation.
- **Cognitive load.** You're working with one structure, but thinking in another.
- **Proof-obligation.** An asymmetric representation commits you to proof-obligations, including recurring symmetry proof-obligations that your symmetry-breaking choices of representative don't affect the final result.

The source of these difficulties is actually our mathematical foundation. We have to roll it right back:

Solution: add symmetry to the foundation

Higher-Order Logic with Atoms = simply-typed λ-calculus over *bool* and *nat* and *atoms*.

Types of HOL with atoms are $\alpha ::= \mathit{bool} \mid \mathit{nat} \mid \mathit{atoms} \mid \alpha \to \alpha$.

Axioms make

- *bool* behave like truth-values and
- *nat* behave like numbers and
- *atoms* behave like an equivariant collection of atoms.

(The axiom for atoms is called Equivariance.)

Foundations with atoms

EZFAC is a ‘sets-flavoured’ version of HOL with atoms.

Intuitively EZFAC is a foundation with:

- Truth-values.
- Numbers: the canonical infinite ordered set.
- Atoms: the canonical infinite unordered set.

Conclusions

Foundations powerfully shape our thinking.

HOL and ZFC are rigid, and this creates practical difficulties representing symmetric data.

HOL with atoms and EZFAC are universes with plentiful symmetries and automorphisms. They are consistent with choice functions, easily support nominal techniques, and should be relatively simple to implement in a theorem-prover.