What is an EUTxO blockchain?

Galois Tech Talk series

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Thanks

Thanks to the team at Galois for the invitation to speak.

This talk is based on joint work with Lars Brünjes.

Implementation

What follows has been implemented in Haskell using the Nominal Datatypes Package:

- Code at: `tinyurl.com/nomeutxo`
- Package at: `tinyurl.com/nominaldata`
- Fetch source: `git clone https://github.com/bellissimogiorno/nominal.git`

See the journal paper (`https://arxiv.org/abs/2007.12404`, submitted), and also a conference paper (`https://arxiv.org/abs/2003.14271`, accepted).

Call these equations idealised EUTxO, and a solution an IEUTxO model or solution

$$
\begin{aligned}
\text{Input} &= \mathbb{A} \times \alpha \\
\text{Validator} &\subseteq \mathit{pow}(\text{Transaction}) \\
\text{Output} &= \mathbb{A} \times \text{Validator} \\
\text{Transaction} &\subseteq [\text{Input}] \times [\text{Output}] \\
\\
\text{Chunk} &\subseteq [\text{Transaction}] \\
\text{Blockchain} &= \{\mathit{ch} \in \text{Chunk} \mid \mathit{utxi}(\mathit{ch}) = \varnothing\}
\end{aligned}
$$

Above, [-] means 'list of -', as per Haskell notation.

You might stare at the 'Blockchain' type, but you should *also* pay attention to the 'Chunk' type. More on that later.

Warning: this figure elides details. See Def 3.1.1 of journal paper.

Base types: α and $\mathbb{A}$

$$\begin{aligned}
\text{Input} &= \mathbb{A} \times \alpha \\
\text{Validator} &\subseteq \mathit{pow}(\text{Transaction}) \\
\text{Output} &= \mathbb{A} \times \text{Validator} \\
\text{Transaction} &\subseteq [\text{Input}] \times [\text{Output}] \\
\text{Chunk} &\subseteq [\text{Transaction}]
\end{aligned}$$

- α is a base data type for our blockchain. Assuming sufficient Gödel encoding & disregarding efficiency, we could take $\alpha = \mathbb{N}$.
- $\mathbb{A}$ is a countably infinite set of location IDs. Each Input will get a unique ID; as will each Output.
  Atoms are atoms of Zermelo-Fraenkel set theory with atoms, as per nominal techniques (relevant to reading the Haskell implementation).

Input, Output, Validator, Transaction

$$\begin{aligned}
\text{Input} &= \mathbb{A} \times \alpha \\
\text{Validator} &\subseteq \mathit{pow}(\text{Transaction}) \\
\text{Output} &= \mathbb{A} \times \text{Validator} \\
\text{Transaction} &\subseteq [\text{Input}] \times [\text{Output}] \\
\text{Chunk} &\subseteq [\text{Transaction}]
\end{aligned}$$

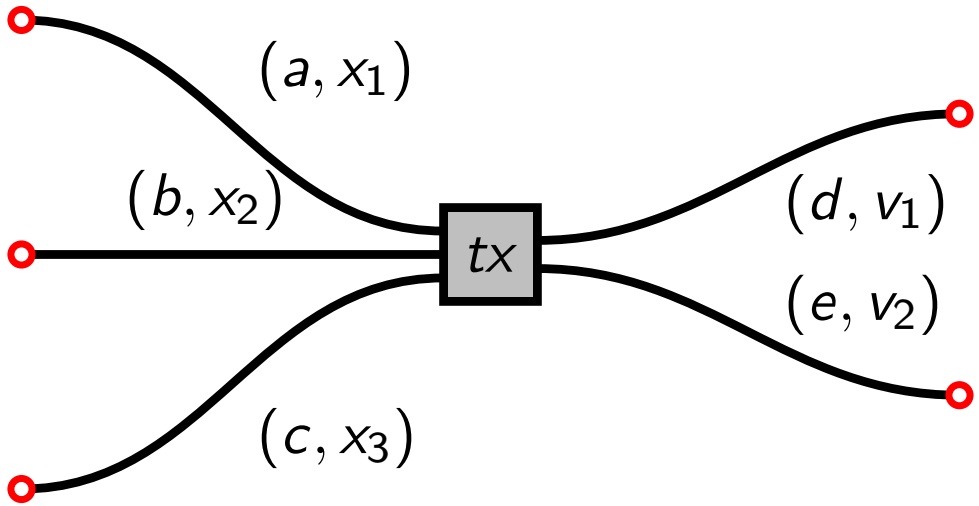
An input is an α, located at some $\mathbb{A}$-position.

A output is a validator, located at some $\mathbb{A}$-position.

A validator specifies a set of 'valid transactions'. We don't take the full powerset — e.g. we expect validity to be computable (also avoids cardinality issues).

A transaction is a list of inputs, and a list of outputs. Also subject to validity constraints (more on this later).

A diagram: a typical transaction



The transaction tx has:

- three inputs $x_1, x_2, x_3 \in \mathbb{N}$, located at $a, b, c \in \mathbb{A}$ respectively, and
- two validators $v_1, v_2 \subseteq$ Transaction, located at $d, e \in \mathbb{A}$ respectively.

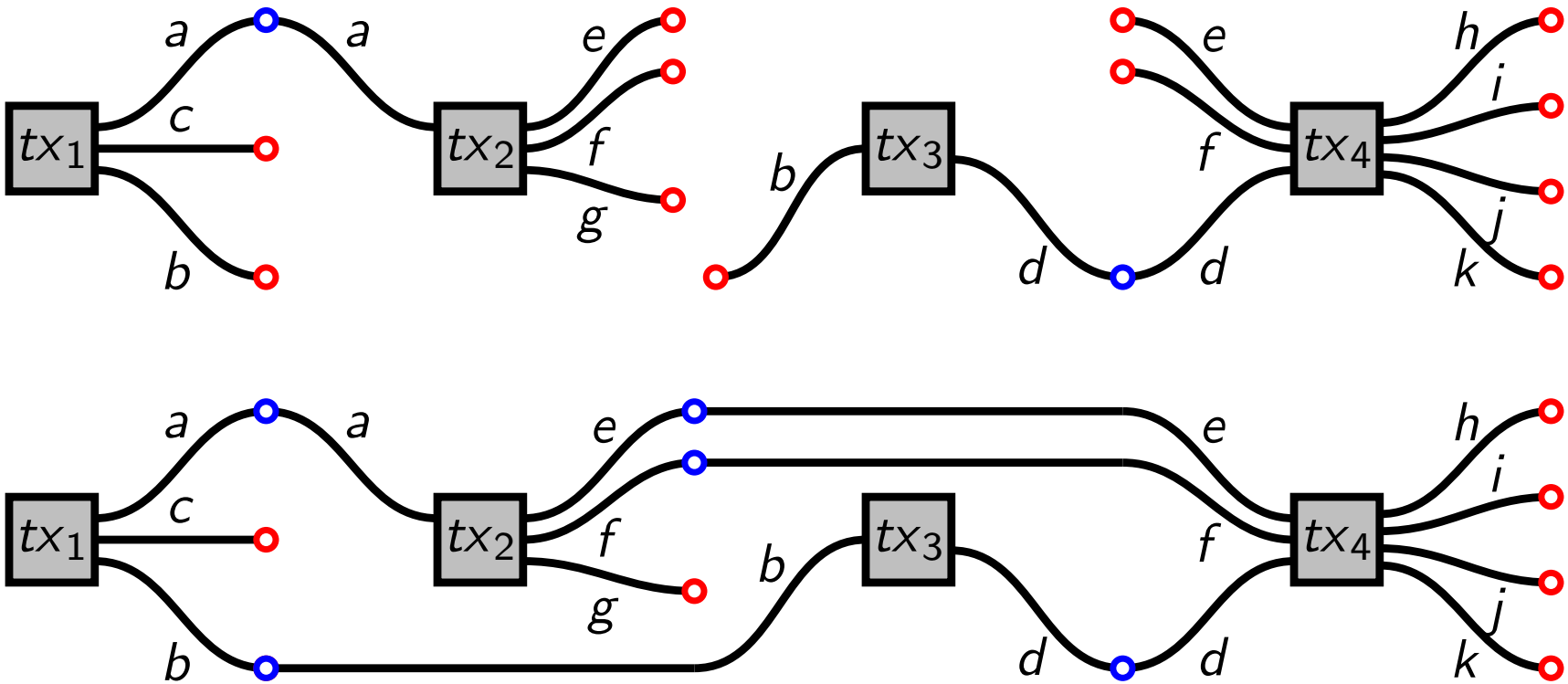
Chunk

$$\begin{aligned}\mathsf{Input} &= \mathbb{A} \times \alpha \\ \mathsf{Output} &\subseteq \mathbb{A} \times \mathit{pow}([\mathsf{Input}] \times [\mathsf{Output}]) \\ \mathsf{Chunk} &\subseteq [\,[\mathsf{Input}] \times [\mathsf{Output}]\,]\end{aligned}$$

A Chunk is a list of transactions (pairs of finite sets of inputs and outputs), subject to validity constraints:

- Each input in a chunk must have a unique position amongst inputs, likewise for outputs.
- If an output shares a position with an input then:
  - the pair must be unique with that position; and
  - occur in the order output-input, so a later input points to at most one earlier output; and
  - an output pointed to must validate the pointing input's transaction, with that input moved to the list head.

Three chunks



UTxIs and UTxOs

Chunks are similar to a datatype of abstract syntax with binding. Positions are reminiscent of π-calculus channels. Chunks have:

- dangling / free / unspent inputs (inputs not bound to an earlier output), and
- dangling / free / unspent outputs (outputs not bound to a later input), and therefore
- α-equivalence on positions of bound output-input pairs.

Call an unspent input a UTxI and an unspent output a UTxO.

A blockchain is a chunk with no free inputs (*left-closed*):

$$\begin{aligned} \text{Chunk} &\subseteq [\text{Transaction}] \\ \text{Blockchain} &= \{ch \in \text{Chunk} \mid \mathit{utxi}(ch) = \varnothing\} \end{aligned}$$

Q. Which of the chunks in the last slide, are blockchains?

Chunks as an algebra

Chunks form a partially-ordered partial monoid.

- The unit is the empty chunk (the chunk consisting of no transactions).
- Composition is list concatenation, subject to validity conditions (no name-clash with positions, no failed validators).
  We may create α-bindings: UTxOs dangling *right* from the *left* chunk may bind to UTxIs dangling *left* from the *right* chunk.
- This partial monoid is partially-ordered by sublist inclusion. It is a fact that validity is preserved by taking sublists.

Operationally as well as mathematically, chunks can be nicer to work with than blockchains.

I discovered this from the Haskell: to be elegant, the code wanted chunks and a partial monoid structure. The maths followed.

In these slides:

- We consider the *structural* aspects of blockchain.
  The challenge of securing these structures cryptographically, is not considered. Yet, at least conceptually, we still provide a clean abstraction to which the crypto side can attach.
- This talk is a mathematical abstraction. Practical implementation is more complex, of course.
  Yet, the type equations and their algebraic structure brings clarity which I (at least) find helpful.
- Even with these elisions, our Haskell implementation demonstrates that this idealisation still has operational content and yields executable code.

What's original?

- The notion of (E)UTxO blockchain is established machinery (Bitcoin; "The extended UTXO model").
- The mathematical idealisation on Slide 4 is new.
- The focus on chunks, their partial monoid structure, and the (minor, but explicit) idea of UTxIs, is new, so far as I am aware.
- Equating names of output-input pairs explicitly with α-equivalence (like in syntax), and applying a nominal model to their operational semantics (cf. the code), is new.
- See the papers for more.

Let's use our maths to express some results

Notation 1. Let variables named ch range over $Chunk$.

Definition 2. Write $pos(ch)$ for the positions of $ch \in$ Chunk. Thus, $pos(ch) = utxi(ch) \cup utxo(ch)$.

Definition 3. Write $ch\#ch'$ when $pos(ch) \cap pos(ch') = \varnothing$.

Remark 4. $ch\#ch'$ is a strong orthogonality assertion. The inputs and outputs of ch and ch' cannot connect, and they can't communicate or compete for UTxOs or UTxIs of other chunks.

Let's use our maths to express some results

Notation 5. Write $ch \bullet ch'$ when $ch \cdot ch'$ is defined.

Lemma 6 (simple). $ch \# ch'$ implies $ch \bullet ch' \wedge ch' \bullet ch$.
Sketch proof: If they don't share positions they can't interact: there can't be name-clash between them, and their validators can't fail on one another, because their validators can't be referenced, because they don't know one another's positions.

Lemma 7 (slightly harder). $ch \bullet ch' \wedge ch' \bullet ch$ implies $ch \# ch'$.
Sketch proof: If an input in ch' points to an output in ch then $\neg(ch \bullet ch')$, because this would violate that an input must point to an earlier output position. So they can't share positions.

We don't develop observational equivalence in these slides, but if we did then using Lemmas 6 and 7 we would identify $ch \# ch'$ with commutativity. See also next Theorem:

More results

Theorem 3.3.2 of journal paper. Suppose $ch \bullet ch_2 \bullet ch_1$. Then:

$$utxi(ch \cdot ch_1) = utxi(ch \cdot ch_2 \cdot ch_1) \Rightarrow ch_1 \# ch_2$$

Technical as this may seem, it is an important purity result.

Consider the special case that UTxIs are $\varnothing$ (so: blockchains), and ch is the current chain. Then *if* we can append ch_1 to the chain now, and we can also append it later (after some ch_2 attaches), then the UTxOs that ch_2 references are necessarily apart from those referenced by inputs of ch_1.

So ch_2 might cause ch_1 to fail to attach to ch, but if it doesn't then it *can't* interact with ch_1; ch_2 might block ch_1 from attaching, but has no effect on ch_1's outcome if successful.

A simple example: it counts!

Take $\alpha = \mathbb{N}$. *(An even simpler possiblity is $\alpha = \{*\}$, but I want non-trivial validators).*

We will now choose subsets:

$$\begin{aligned} \mathsf{Input} &= \mathbb{A} \times \mathbb{N} \\ \mathsf{Validator} &\subseteq \mathit{pow}(\mathsf{Transaction}) \\ \mathsf{Output} &= \mathbb{A} \times \mathsf{Validator} \\ \mathsf{Transaction} &\subseteq [\mathsf{Input}] \times [\mathsf{Output}] \\ \mathsf{Chunk} &\subseteq [\mathsf{Transaction}] \end{aligned}$$

A simple example: it counts!

We admit transactions $\mathsf{Transaction} \subseteq [\mathsf{Input}] \times [\mathsf{Output}]$ of the form

$$succ_{i,p,p'} = \Big(\ [(p,i)], \ [(p', val_{p',i'})] \ \Big)$$
$$\text{where} \quad val_{p',i'} = ([(p',i')], \_) \longmapsto i' = i{+}1$$

for $i \in \mathbb{N}$ and $p, p' \in \mathbb{A}$ distinct. That's a singleton input $[(p,i)]$ and a singleton output $[(p', val_{p',i'})]$, where val validates a transaction iff its input points to p' and carries $i{+}1$. Thus:

$$\mathsf{Transaction} = \Big\{ succ_{i,p,p'} \mid i \in \mathbb{N}, \ p{\neq}p' \in \mathbb{A} \Big\}.$$

Admit any chunk, if positions match up and validators are satisfied (in particular, at most one UTxI and UTxO). So:

- Composition is list concatenation; and
- the unit is the empty chunk [].

A simple example: it counts!

Proposition 9. There is a homomorphism of partial monoids from $(\text{Chunk}, [], \cdot)$ to $(\mathbb{N}, 0, +)$, given by mapping a chunk to its length as a transaction-list:

- The empty chunk maps to 0.
- Composition — attaching an n-chunk to an n'-chunk — maps to addition $n + n'$.

Thus, our example counts; each transaction is visibly a 'successor' operation, subject to solving the puzzle of knowing the position of the end UTxO of the left-hand n-chunk, and knowing its final value. That's fine: we expect partiality and this is just part of the 'crypto' aspect of the model.

Note: Proposition 9 holds for any chunk system (not only this example).

A simple example

The previous model doesn't have any blockchains (left-closed chunks), because we did not admit a genesis block (a transaction without inputs).

I don't see this as a problem — what system doesn't allow users to download partial blockchains nowadays? — but it's also easily fixed. Admit zero transactions

$$0_p = ([], [(p,\ ([(p, i)], \_) \mapsto i = 0)])$$

and admit a chunk provided it contains *at most* one zero transaction.

A simple example

Is this a simple example? Yes! But I propose that more complex examples are, mathematically, just fancy-pants versions of this one.

That is not to say that blockchains are simple, nor that we have a accounted for all complexity and extensions; quite the contrary.

But what we have done is simplify complexity, abstract detail, and obtain a clear model to guide us.

Example: restoring some complexity, to see how it's done

In practice, Validator $\subseteq$ *pow*(Transaction) is supplied as a script (not direclty as a set) — in the literature, validators are taken to *be* scripts, which makes perfect operational sense.

A script is propagated along outputs and should not necessarily change with every transaction. To reflect this in the maths we just require an additional type parameter β:

$$\begin{aligned} \text{Input} &= \mathbb{A} \times \alpha \\ \text{Validator} &\subseteq \mathit{pow}(\beta \times \text{Transaction}) \\ \text{Output} &= \mathbb{A} \times \beta \times \text{Validator} \\ \text{Transaction} &\subseteq [\text{Input}] \times [\text{Output}] \\ \text{Chunk} &\subseteq [\text{Transaction}] \end{aligned}$$

The β in Output tells us which part of $v \in$ Validator (thought of as a script with a β parameter) to reference.

Conclusions

We've seen a simple, abstract presentation of the (E)UTxO model and sketched its properties.

We noted that *chunks* have algebraic properties, and form a partially-ordered partial monoid with (π-calculus-like) channels.

Name-binding corresponds to linking UTxOs to UTxIs and this connection is non-superficial in the sense that notions of support and apartness $\#$ correspond to commutativity and equivalence properties of chunks.

Note: a compact 'mathematics of blockchains' is possible.

It leads to mathematical theorems, testable properties, and structures the system in a way which (at least for me) is helpful. And it's quite elegant.

Conclusions

A journal paper is here:

http://gabbay.org.uk/papers.html#whaeb

https://arxiv.org/abs/2007.12404

(IEUTxO are page 5 onwards; the corresponding algebras are page 15 onwards.)

The package is here:

https://github.com/bellissimogiorno/nominal

The journal paper has further material, including an algebra-style axiomatisation of IEUTxO solutions as *abstract chunk systems*.

An example of future work would be to apply a similar analysis to an accounts-style system (Ethereum vs. Bitcoin).

Thanks for listening.