Stone duality for nominal Boolean algebras with 'new': topologising Banonas

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Abstract. We define Boolean algebras over nominal sets with a function-symbol и mirroring the И 'fresh name' quantifier. We also define dual notions of nominal topology and Stone space, prove a representation theorem over fields of nominal sets, and extend this to a Stone duality.

1 Introduction

$\text{И}a.\phi(a)$ means 'for *fresh* a, $\phi(a)$ holds'. Here a is intended to be an atomic resource: examples are a variable symbol, memory location, channel name, or (in security) nonce.

И is interesting from both a mathematical and computational point of view. It is *self-dual* ($\neg\text{И}a.\phi(a) \Leftrightarrow \text{И}a.\neg\phi(a)$) and this gives a computationally useful *some/any property*: to prove И we check $\phi(a)$ for *some* fresh name, but we can then use *any* fresh b. (Definitions below or in [18,14].)

И also appears to be useful. It has been applied to syntax-with-binding [18], resource generation in functional programming [29], local names in incomplete trees [5], π-calculus name-restriction [3], game semantics [1,33], and domains [32]. See Example 3.6 for more discussion.

Proof-theories for И have been suggested [11,15,7,10], but no equational axiomatisation or representation theorem. Nominal domains have been studied, but not topologies. As we shall see, algebra and topology with И are non-trivial and interesting.

In this paper we study Boolean logic with И in nominal sets. We introduce Banonas (Boolean Algebra, NOminal with Иa) and nominal Stone spaces, and prove representation and duality theorems.

Stone's representation does not generalise easily to nominal sets: the ultrafilter lemma breaks down (since a union of finitely-supported sets need not be finitely-supported). To 'fix' this, a name-restriction operation seems to be *required*. So Banonas seem to arise *naturally* from the proofs: the mathematics leads us to this structure even if we did not start off with it. On that topic, Banonas are also Boolean algebra objects in the category *Res* of *nominal restriction sets* considered recently in [26,27]. There seems to be convergence in the recent literature, on name-restriction as an interesting object of study.

One application of Stone duality relates coalgebras for 'sufficently well-behaved' functors on sets, with algebras for 'sufficently well-behaved' functors on Boolean algebras (definitions in [21]). This allows a uniform construction of logic(s) for coalgebras. So this paper lays the groundwork for a duality-based approach to *nominal coalgebraic logic* with built-in name generation. Coalgebras on nominal sets do provide a natural semantics for name-passing calculi such as π-calculus (see e.g. [9, Section 1.2] and [30] for information and references). See Example 3.6 and the Conclusions for more discussion of applications.

The results in this paper describe a natural mathematical building-block of inherent mathematical elegance, which seems to arise spontaneously by considering algebra and topology in a 'nominal' setting, and with potential practical interest. We discuss further work in the Conclusions.

2 Basic definitions for nominal sets

A nominal set is a 'set with names'. The notion of a name being 'in' an element is given by support $supp(x)$ (Definition 2.7). For more details of nominal sets, see [18,14]. The reader can reasonably think of $supp(x)$ as 'the free atoms of x', but without committing to x being syntax—on the contrary, x could have any structure provided it admits a permutation action, and this includes taking powersets as we see later in Section 5.

Definition 2.1. Fix a countably infinite set of **atoms** $\mathbb{A}$. We use a **permutative convention** that $a, b, c, \dots$ range over *distinct* atoms.

Definition 2.2. A **(finite) permutation** π is a bijection on atoms such that $nontriv(\pi) = \{a \mid \pi(a) \neq a\}$ is finite.

Write id for the **identity** permutation such that $\mathrm{id}(a) = a$ for all a. Write $\pi' \circ \pi$ for composition, so that $(\pi' \circ \pi)(a) = \pi'(\pi(a))$. Write $\pi^{\text{-}1}$ for inverse, so that $\pi^{\text{-}1} \circ \pi = \mathrm{id} = \pi \circ \pi^{\text{-}1}$. Write $(a\ b)$ for the **swapping** (terminology from [18]) mapping a to b, b to a, and all other c to themselves, and take $(a\ a) = \mathrm{id}$.

Definition 2.3. If $A \subseteq \mathbb{A}$ define $fix(A) = \{\pi \mid \forall a \in A.\pi(a) = a\}$.

Definition 2.4. A **set with a permutation action** X is a pair $(|\mathsf{X}|, \cdot)$ of an **underlying set** $|\mathsf{X}|$ and a **permutation action** written $\pi{\cdot}x$ which is a group action on $|\mathsf{X}|$, so that $\mathrm{id}{\cdot}x = x$ and $\pi{\cdot}(\pi'{\cdot}x) = (\pi \circ \pi'){\cdot}x$ for all $x \in |\mathsf{X}|$ and permutations π and π'.

Say that $A \subseteq \mathbb{A}$ **supports** $x \in |\mathsf{X}|$ when $\forall \pi. \pi \in fix(A) \Rightarrow \pi{\cdot}x = x$. If a finite A supporting x exists, call x **finitely-supported**.

Definition 2.5. Call a set with a permutation action X a **nominal set** when every $x \in |\mathsf{X}|$ has finite support. X, Y, Z will range over nominal sets.

Definition 2.6. Call a function $f \in |\mathsf{X}| \to |\mathsf{Y}|$ **equivariant** when $\pi{\cdot}f(x) = f(\pi{\cdot}x)$ for all permutations π and $x \in |\mathsf{X}|$. In this case write $f : \mathsf{X} \to \mathsf{Y}$.

Definition 2.7. Suppose X is a nominal set and $x \in |\mathsf{X}|$. Define the **support** of x by $supp(x) = \bigcap\{A \mid A \text{ supports } x\}$. Write $a\#x$ as shorthand for $a \notin supp(x)$ and read this as a is **fresh for** x.

(**Commute**)		$x \wedge y = y \wedge x$
(**Assoc**)		$(x \wedge y) \wedge z = x \wedge (y \wedge z)$
(**Huntington**)		$x = \neg(\neg x \wedge \neg y) \wedge \neg(\neg x \wedge y)$
(**Swap**)		$\mathsf{И}a.\mathsf{И}b.x = \mathsf{И}b.\mathsf{И}a.x$
(**Garbage**)	$a\#x \Rightarrow$	$\mathsf{И}a.x = x$
(**Distrib**)		$\mathsf{И}a.(x \wedge y) = (\mathsf{И}a.x) \wedge (\mathsf{И}a.y)$
(**SelfDual**)		$\neg\mathsf{И}a.x = \mathsf{И}a.\neg x$
(**Alpha**)	$b\#x \Rightarrow$	$\mathsf{И}a.x = \mathsf{И}b.(b\ a)\cdot x$

Fig. 1: Axioms of Banonas.

Theorem 2.8. *Suppose* X *is a nominal set and* $x \in |\mathsf{X}|$. *Then* $supp(x)$ *is the unique least finite set of atoms that supports* x. *(Proofs in [18,14].)*

Definition 2.9. Write $\pi|_A$ for the partial function which is π restricted to A.

Corollary 2.10. *1. If* $\pi(a) = a$ *for all* $a \in supp(x)$ *then* $\pi\cdot x = x$.
2. If $\pi|_{supp(x)} = \pi'|_{supp(x)}$ *then* $\pi\cdot x = \pi'\cdot x$.
3. $a\#x$ *if and only if* $\exists b.b\#x \wedge (b\ a)\cdot x = x$.

Proposition 2.11. $supp(\pi\cdot x) = \{\pi(a) \mid a \in supp(x)\}$ *(cf. Definition 5.1).*

3 Nominal Boolean algebra with И

Definition 3.1. A **nominal Boolean algebra with** И (with a new-binder) or **Banona** is a tuple $(\mathsf{B}, \wedge, \neg, \mathsf{И})$ of a nonempty nominal set B, and equivariant functions (Definition 2.6)

- *conjunction* $\wedge : \mathsf{B} \times \mathsf{B} \to \mathsf{B}$ written $x \wedge y$ (for $\wedge(x, y)$),
- *negation* $\neg : \mathsf{B} \to \mathsf{B}$ written $\neg x$, and
- *new* $\mathsf{И} : \mathbb{A} \times \mathsf{B} \to \mathsf{B}$ written $\mathsf{И}a.x$ (for $\mathsf{И}(a, x)$),

such that the equalities in Figure 1 hold.

Banonas arise naturally as nominal powersets (Section 5), just as Boolean algebras arise naturally as 'ordinary' powersets. For an example consider any (for simplicity) unary predicate P on some syntax, represented as the set of terms of which it is true. Then asserting that P holds 'if a is fresh' (as in 'if a is fresh then for all u, $t[a{::=}u] = t$) is an instance of $\mathsf{И}a.P$.

Remark 3.2. (**Commute**), (**Assoc**), and (**Huntington**) axiomatise Boolean algebra (see, e.g., [25]). We write $\bot$ for $x \wedge \neg x$, $x \vee y$ for $\neg(\neg x \wedge \neg y)$, and $x \leq y$ for $x \wedge y = x$. With these definitions, the axioms ensure standard properties of Boolean algebra including: absorption, distributivity, and poset properties.

И is a name-restriction operation in the sense of [26, Definition 1] since it satisfies (**Swap**), (**Garbage**), and (**Alpha**). So each Banona has an underlying nominal restriction set. (**Distrib**), (**SelfDual**) imply that $\wedge$ and $\neg$ are morphisms in the category *Res* [27], thus Banonas are Boolean algebra objects in *Res*.

All these axioms are valid properties of the И-quantifier [18] (Definition 4.3, in this paper). See Theorem 5.9 for a proof.[1]

Lemma 3.3. *If $x \leq y$ then $\text{и}b.x \leq \text{и}b.y$. In words: и is monotone.*

Proof. By $x \leq y$ we mean $x \wedge y = x$. So $\text{и}b.(x \wedge y) = \text{и}b.x$. We use (**Distrib**).

Lemma 3.4. *$a\#\text{и}a.x$. As a corollary, $\text{и}a.\text{и}a.x = \text{и}a.x$.*

Proof. Choose a fresh b (so $b\#x$). By (**Alpha**) $\text{и}a.x = \text{и}b.(b\ a)\cdot x$. By equivariance $\text{и}b.(b\ a)\cdot x = (b\ a)\cdot\text{и}a.x$. By Proposition 2.11 $a\#(b\ a)\cdot\text{и}a.x$. The result follows. The corollary follows using (**Garbage**).

Definition 3.5. Call a function $f \in |\mathsf{B}'| \to |\mathsf{B}|$ a (Banona) **morphism** when:

$$f(x \wedge y) = f(x) \wedge f(y) \quad f(\neg x) = \neg f(x) \quad f(\text{и}a.x) = \text{и}a.f(x) \quad f(\pi\cdot x) = \pi\cdot f(x)$$

Write BAnona for the category of Banonas and Banona morphisms.

Example 3.6.
- Call a set $X \subseteq \mathbb{A}$ **cofinite** when $\mathbb{A} \setminus X$ is finite. The set of finite or cofinite sets of atoms is a Banona where conjunction and negation are interpreted as intersection and complement respectively, and $\text{и}a.X = X \setminus \{a\}$ if X is finite, and $\text{и}a.X = X \cup \{a\}$ if X is cofinite (cf. Example 5.3). In Section 5 we exhibit a class of Banonas given by nominal powersets; The results in Section 6 prove this class complete for all possible examples in a certain formal sense.
- The discrete *nominal restriction set* $\mathbb{B} = \{\mathit{True}, \mathit{False}\}$ from [27, Section 3.2] is also a Banona.
- The term model for nominal logic constructed by Cheney and Urban in [8] is a Banona where we interpret conjunction as conjunction, negation as negation, and $\text{и}a.\phi$ as 'for fresh a, ϕ' as specified in Figure 9 of [8].
- Predicates of Cardelli and Gordon's *ambient calculus* up to logical equivalence form a (very richly-structured) Banona where и is interpreted as И as defined in Definition 4.3 of [6] (see [6, Corollary 4.5]).
- Formulas of hybrid logic with downarrow $\downarrow x.\phi$ (see, e.g, [2]) up to logical equivalence, where atoms serve as *world variables*, permutation is syntactic permutation and freshness is 'not free in', are a Banona where и maps to $\downarrow$. A broader class of Banonas (with additional operators) can be constructed by adapting an abstract equational class of algebras including sets denotations for hybrid logic defined in [22] to nominal setting.

[1] The axioms are *nominal* algebraic because of their freshness side-conditions. In a sense which has been made formal, nominal algebra is an equational logic [17,12,20], and is sound and complete for nominal sets.

4 Equivariance and the И-quantifier

Our reasoning can be formalised in Zermelo-Fraenkel set theory with atoms (**ZFA**). Nominal sets can be implemented in ZFA sets such that nominal sets map to equivariant elements (elements with empty support) and the permutation action maps to 'real' permutation of atoms in the model [14, Subsection 9.3]. From this, follow short and elegant proofs of theorems about equivariance and support. For more details, see [14, Section 4].

Definition 4.1. If $\overline{x}$ denotes a list $x_1, \ldots, x_n$, write $\pi{\cdot}\overline{x}$ for $\pi{\cdot}x_1, \ldots, \pi{\cdot}x_n$.

Theorem 4.2. *Suppose $\phi(\overline{x})$ is a ZFA predicate on variables included in $\overline{x}$. Suppose $\chi(\overline{x})$ is a ZFA function specified using a list of variables included in $\overline{x}$. Then we have the following principles:*

1. **Equivariance of predicates.** $\phi(\overline{x}) \Leftrightarrow \phi(\pi{\cdot}\overline{x})$.[2]
2. **Equivariance of functions.** $\pi{\cdot}\chi(\overline{x}) = \chi(\pi{\cdot}\overline{x})$.
3. **Conservation of support.** *If $\overline{x}$ denotes elements with finite support then $supp(\chi(\overline{x})) \subseteq supp(x_1) \cup \cdots \cup supp(x_n)$.*

Definition 4.3. Write $\text{И}a.\phi(a)$ for '$\{a \mid \neg\phi(a)\}$ is finite'.

Remark 4.4. We can read И as 'for all but finitely many a', 'for fresh a', or 'for new a'. This is like a 'for most' quantifier [35], and is a *generalised quantifier* [19, Section 1.2.1]. However, И over nominal sets satisfies additional properties:

Theorem 4.5. *Suppose $\phi(\overline{z}, a)$ is a predicate on variables included in $\overline{z}, a$.*[3] *Suppose $\overline{z}$ denotes elements with finite support. Then the following are equivalent:*

$$\forall a.(a \in \mathbb{A} \wedge a\#\overline{z}) \Rightarrow \phi(\overline{z}, a) \qquad \text{И}a.\phi(\overline{z}, a) \qquad \exists a.a \in \mathbb{A} \wedge a\#\overline{z} \wedge \phi(\overline{z}, a)$$

This is the *some/any property*: to prove a И-quantified property we test it for *one* fresh atom; we may then use it for *any* fresh atom.

5 The nominal powerset as an algebra

Definition 5.1. Define the **nominal powerset** $pow(\mathsf{X})$ by:

- $U \subseteq |\mathsf{X}|$ has the **pointwise** action $\pi{\cdot}U = \{\pi{\cdot}x \mid x \in U\}$.
- $|pow(\mathsf{X})|$ is the set of finitely-supported $U \subseteq |\mathsf{X}|$.

[2] $\overline{x}$ must contain *all* the variables mentioned in the predicate. It is not the case that $a = a$ if and only if $a = b$—but it is the case that $a = b$ if and only if $b = a$.

[3] ϕ has to be expressible in the language of ZFA set theory without choice. Every ϕ used in this paper will satisfy this property.

Definition 5.2. If $X \in |pow(\mathsf{X})|$ then define $\mathsf{n}a.X = \{x \mid \text{И}b.(b\ a)\cdot x \in X\}$.

Example 5.3. $|pow(\mathbb{A})|$ is the set of finite and cofinite subsets of atoms and $\mathsf{n}a.X$ is characterised on $pow(\mathbb{A})$ by:

- If X is finite then $\mathsf{n}a.X = X \setminus \{a\}$.
- If X is cofinite then $\mathsf{n}a.X = X \cup \{a\}$.

In the case of $X \in |pow(\mathsf{X})|$ for general X, the elements added/removed by n are called *crucial elements* in [13, Subsection 4.2] and correspond to adding/removing elements from a-orbits [13, Subsection 4.1]. In fact $\mathsf{n}a.$ is equal to the operation $X{-}a$ described in [13, Subsection 4.2], used there for different purposes.

Remark 5.4. Definition 5.2 is the sets-based interpretation of И which we will prove sound and complete for the axioms in Definition 3.1. Visibly, n is defined using И. Conversely, $\text{И}a.\phi(a)$ if and only if $a \in \mathsf{n}a.\{x \in \mathbb{A} \mid \phi(x)\}$.

Lemma 5.5. *If $X \in |pow(\mathsf{X})|$ then $\mathsf{n}a.X \in |pow(\mathsf{X})|$.*

Proof. By Theorem 4.2 $supp(\mathsf{n}a.X) \subseteq supp(X) \cup \{a\}$.

Lemma 5.6. *1. If $a\#X$ then $\mathsf{n}a.X = X$.*
2. $\mathsf{n}a.(X \cap Y) = (\mathsf{n}a.X) \cap (\mathsf{n}a.Y)$.
3. $\mathsf{n}a.(|\mathsf{X}| \setminus X) = |\mathsf{X}| \setminus \mathsf{n}a.X$.
As a corollary, if $X \subseteq Y$ then $\mathsf{n}a.X \subseteq \mathsf{n}a.Y$.

Proof (Sketch). Suppose $a\#X$. Choose $x \in |\mathsf{X}|$ and fresh b (so $b\#x, X$). By the pointwise action and Corollary 2.10 $(b\ a)\cdot x \in X$ if and only if $x \in X$.

Choose $x \in |\mathsf{X}|$ and fresh b (so $b\#x, X, Y$). Then $(b\ a)\cdot x \in X \cap Y$ if and only if $(b\ a)\cdot x \in X$ and $(b\ a)\cdot x \in Y$.

Choose $x \in |\mathsf{X}|$ and fresh b (so $b\#x, X, |\mathsf{X}| \setminus X$). Then $(b\ a)\cdot x \in |\mathsf{X}| \setminus X$ if and only if $(b\ a)\cdot x \notin X$.

Lemma 5.7. *$b\#X$ implies $\mathsf{n}a.X = \mathsf{n}b.(b\ a)\cdot X$. As a corollary, $a\#\mathsf{n}a.X$.*

Proof (Sketch). Choose fresh b (so $b\#X$) and $x \in |\mathsf{X}|$. By the pointwise action $(b\ c)\cdot x \in (a\ b)\cdot X$ if and only if $x \in (b\ c)\cdot (a\ b)\cdot X$. By Corollary 2.10 $(b\ c)\cdot (a\ b)\cdot X = (a\ c)\cdot X$. The result follows.

Lemma 5.8. *$\mathsf{n}a.\mathsf{n}b.X = \mathsf{n}b.\mathsf{n}a.X$.*

Proof. By routine calculations using the fact that $(a'\ a)\cdot (b'\ b)\cdot x = (b'\ b)\cdot (a'\ a)\cdot x$.

Theorem 5.9. *$(pow(\mathsf{X}), \cap, |\mathsf{X}|\setminus\text{-}, \mathsf{n})$ is an object of* BAnona.

Proof. Validity of (**Commute**), (**Assoc**), and (**Huntington**) is by routine sets calculations. Validity of (**Alpha**) and (**Swap**) is by Lemmas 5.7 and 5.8. Validity of (**Garbage**), (**Distrib**), and (**SelfDual**) is by Lemma 5.6.

Proposition 5.10. *If $a\#x$ then $x \in X$ if and only if $x \in \mathsf{n}a.X$.*

Proof. Choose b fresh (so $b\#x, X$). By definition $x \in \mathsf{n}a.X$ when $(b\ a)\cdot x \in X$. By Corollary 2.10 $(b\ a)\cdot x = x$. The result follows.

6 A representation theorem

We introduce n-*filters* of a Banona B (Definition 6.1). We define the *canonical extension* $\mathsf{B}^\bullet$ as the nominal powerset of its maximal n-filters (Definition 6.20). Finally we prove B isomorphic to a subalgebra of $\mathsf{B}^\bullet$ (Theorem 6.25).

6.1 n-Filters

Definition 6.1. An **n-filter** is a finitely-supported subset $p \subseteq |\mathsf{B}|$ such that:

$$1.\ \bot \notin p \qquad 2.\ \forall x, y.(x \in p \wedge y \in p) \Leftrightarrow (x \wedge y \in p) \qquad 3.\ \text{И}a.\forall x.x{\in}p \Rightarrow \mathsf{n}a.x{\in}p$$

Remark 6.2. The first two conditions of Definition 6.1 are as standard. The third condition corresponds to a property of nominal sets (Proposition 5.10) in a sense made formal in the proof of Lemma 6.12.

Definition 6.3. Given $C \subseteq \mathbb{A}$ and $x \in |\mathsf{B}|$ define $\mathsf{n}C.x$ by:

$$\mathsf{n}C.x = \mathsf{n}c_1 \ldots \mathsf{n}c_n.x \quad \text{where} \quad C \cap supp(x) = \{c_1, \ldots, c_n\}$$

By assumption x has finite support so $C \cap supp(x)$ is finite even if C is not. Also, by (**Swap**) the order of the c_i does not matter.

Lemma 6.4. *Suppose $C \subseteq \mathbb{A}$ and $a \in C$. Then:*

$$1.\ \ \mathsf{n}C.\bot = \bot \qquad 2.\ \ \mathsf{n}C.(x \wedge y) = (\mathsf{n}C.x) \wedge \mathsf{n}C.y \qquad 3.\ \ \mathsf{n}C.x = \mathsf{n}C.\mathsf{n}a.x$$

Proof. 1. Follows from (**Garbage**). 2. Follows from (**Distrib**),(**Swap**) and (**Garbage**). 3. If $a\#x$ then $x = \mathsf{n}a.x$ by Lemma 3.4. If $a \in supp(x)$, then $a \in C \cap supp(x)$ and the result follows from the second part of Lemma 3.4 and (**Swap**).

Definition 6.5. Suppose $z \in |\mathsf{B}|$. Write $C = \mathbb{A} \setminus supp(z)$. Define $z{\uparrow}$ by

$$z{\uparrow} = \{x \mid z \leq \mathsf{n}C.x\}.$$

Remark 6.6. The standard definition of $z{\uparrow}$ is $\{x \mid z \leq x\}$. This definition seems to *not work*, and proofs based on it break. We can view Definition 6.5 as elaborating the standard definition to respect property 3 of Definition 6.1.

Lemma 6.7. *If $z{\in}|\mathsf{B}|$ and $z \neq \bot$ then $supp(z{\uparrow}) \subseteq supp(z)$ and $z{\uparrow}$ is an n-filter.*

Proof. Write $C = \mathbb{A} \setminus supp(z)$. The first part is by Theorem 4.2 and the fact that $supp(C) = supp(z)$.[4] That $z{\uparrow}$ is an n-filter follows from Lemma 6.4.

Definition 6.8. Call an n-filter $p \subseteq |\mathsf{B}|$ **maximal** when for all n-filters $p' \subseteq |\mathsf{B}|$ if $p \subseteq p'$ then $p' = p$.

[4] In fact it can be proved by a further calculation that $supp(z{\uparrow}) = supp(z)$.

We now show that every n-filter is contained in a maximal n-filter (Theorem 6.17). We use Zorn's lemma, but this requires a bound on support. Thus we consider n-filters maximal amongst n-filters with smaller support. Surprisingly, these n-filters are the maximal n-filters (Lemma 6.13).

Lemma 6.9. *Suppose q is an n-filter and suppose $x \notin q$. Write q' for the set $q' = \{z \mid z \vee x \in q\}$. Then q' is an n-filter.*

Proof. That $\bot \notin q'$ and $\forall z_1, z_2.(z_1 \in q' \wedge z_2 \in q') \Leftrightarrow (z_1 \wedge z_2 \in q')$ is routine. $\text{И}a.\forall z.(z \in q' \Rightarrow \text{и}a.z \in q')$ follows from the fact that if a is fresh (so $a\#q, x$) then by (**Garbage**), (**SelfDual**), and (**Distrib**), $\text{и}a.(z \vee x) = (\text{и}a.z) \vee \text{и}a.x = (\text{и}a.z) \vee x$.

Proposition 6.10. *q is maximal if and only if $\forall x.\neg x \in q \Leftrightarrow x \notin q$.*

Proof. For q maximal, suppose $\neg x \notin q$ and $x \notin q$. By Lemma 6.9, $q' = \{z \mid z \vee x \in q\}$ is an n-filter. Also, $\neg x \in q'$ whereas $\neg x \notin q$, so $q \subsetneq q'$, contradicting maximality of q. The rest follows from conditions 1 and 2 of Definition 6.1.

Lemma 6.11. *If q is an n-filter and $C = \mathbb{A} \setminus supp(q)$ then so is $q' = \{z \mid \text{и}C.z \in q\}$.*

Proof. Follows from Lemma 6.4.

Lemma 6.12. *Suppose q is an n-filter such that for all n-filters q', $q \subseteq q'$ and $supp(q') \subseteq supp(q)$ imply $q = q'$. Then $\text{И}a.(\forall x.x \in q \Leftrightarrow \text{и}a.x \in q)$.*

Proof. The left-to-right implication is condition 3 of Definition 6.1. Write $C = \mathbb{A} \setminus supp(q)$. The set $q' = \{z \mid \text{и}C.z \in q\}$ (Definition 6.3) is an n-filter by Lemma 6.11. By condition 3 of Definition 6.1 $q \subseteq q'$, by Theorem 4.2 $supp(q') \subseteq supp(q)$, hence $q' = q$ (note that $supp(C) = supp(q)$). The right-to-left implication follows.

Lemma 6.13. *q is a maximal n-filter if and only if for all n-filters q', $q \subseteq q'$ and $supp(q') \subseteq supp(q)$ imply $q = q'$.*

Proof. The left-to-right implication is trivial. Now suppose q is maximal amongst n-filters with smaller support. By Proposition 6.10 it suffices to show that $\neg x \in q$ if and only if $x \notin q$.

$\neg x \in q$ and $x \in q$ is impossible by conditions 1 and 2 of Definition 6.1.

Now suppose $\neg x \notin q$ and also $x \notin q$. Write $C = \mathbb{A} \setminus supp(q)$ and $x' = \text{и}C.x$. By definition, (**SelfDual**), and Lemma 6.12 $x' \notin q$ and $\neg x' \notin q$.

By Lemma 6.9, $q' = \{z \mid z \vee x' \in q\}$ is an n-filter. Also, $\neg x' \in q'$ whereas $\neg x' \notin q$, so $q \subsetneq q'$. By Theorem 4.2 $supp(q') \subseteq supp(q)$. This contradicts the existence of x.

Definition 6.14. Given a nominal set X, call $Y \subseteq |\mathsf{X}|$ **bounded-supported** when $\bigcup\{supp(x) \mid x \in Y\}$ is finite.

Remark 6.15. By [14, Theorem 2.29] if Y is bounded-supported then Y is finitely-supported and $supp(Y) = \bigcup\{supp(x) \mid x \in Y\}$. See also [32, Definition 3.4.2.3] and subsequent discussion.

Lemma 6.16. *Consider a nominal set* X *and* $\leq$ *a partial order on* X *(*$\leq$ *need not be equivariant, but when we use this in Theorem 6.17, it will be). If every chain* $C \in pow(\mathsf{X})$ *has an upper bound* $b(C)$ *with* $supp(b(C)) \subseteq supp(C)$ *then for every* $x \in |\mathsf{X}|$ *the set* $x^\circ = \{y \in |\mathsf{X}| \mid y \geq x,\ supp(y) \subseteq supp(x)\}$ *has a maximal element.*

Proof. By Remark 6.15 every chain C in x° is finitely-supported and $supp(C) \subseteq supp(x)$. Then C has an upper bound $b(C)$ such that $supp(b(C)) \subseteq supp(C)$. Thus $b(C) \in x^\circ$. The result follows using Zorn's lemma for x°.

Theorem 6.17. *For every* n*-filter* p*, there is a maximal* n*-filter* q *with* $p \subseteq q$*.*

Proof. If C is a finitely-supported chain in the nominal set of n-filters on B ordered by subset inclusion, then an upper bound for C is defined by $\bigcup C = \{x \mid \exists p' \in C. x \in p'\}$, and by Theorem 4.2 $supp(\bigcup C) \subseteq supp(C)$. By Lemma 6.16 the set p° of n-filters p' such that $p \subseteq p'$ and $supp(p') \subseteq supp(p)$ has a maximal element q with respect to inclusion.

Then q is a maximal amongst n-filters with smaller support and $p \subseteq q$. Indeed, if q' is such that $q \subseteq q'$ and $supp(q') \subseteq supp(q)$ then $q' \in p^\circ$, hence $q = q'$. By Lemma 6.13 q is a maximal n-filter (Definition 6.8).

6.2 The canonical extension $\text{-}^\bullet$

Definition 6.18. Define $points(\mathsf{B}) = \{p \subseteq |\mathsf{B}| \mid p \text{ is a maximal } \mathsf{n}\text{-filter}\}$.

Lemma 6.19. $points(\mathsf{B})$ *is a nominal set.*

Proof. By Theorem 4.2 the predicate 'p is a maximal n-filter' holds if and only if the predicate '$\pi{\cdot}p$ is a maximal n-filter' holds.

Definition 6.20. Define the **canonical extension** $\mathsf{B}^\bullet = (pow(points(\mathsf{B})), \wedge, \neg, \text{и})$:

$$A \wedge B = A \cap B \qquad \neg A = |\mathsf{B}^\bullet| \setminus A \qquad \text{и}a.A = \mathsf{n}a.A$$

A and B will range over elements of $|\mathsf{B}^\bullet|$. ($\mathsf{n}a.A$ defined in Definition 5.2.)

Proposition 6.21. $\mathsf{B}^\bullet$ *is a nominal Boolean algebra with* и*.*

Proof. From Theorem 5.9.

Definition 6.22. Define a map $\text{-}^\bullet \in |\mathsf{B}| \to |\mathsf{B}^\bullet|$ by:

$$x^\bullet = \{p \in |points(\mathsf{B})| \mid x \in p\}$$

We need to check that $\text{-}^\bullet$ does map to $|\mathsf{B}^\bullet|$. It suffices to show that $supp(x^\bullet) \subseteq supp(x)$. This follows by Theorem 4.2.

Proposition 6.23. $\text{-}^{\bullet}$ *is an arrow in* BAnona *(Definition 3.5).*

Proof. Equivariance is by Theorem 4.2.

1. $(x \wedge y)^{\bullet} = \{p \in |points(\mathsf{B})| \mid x \wedge y \in p\}$. By assumption in Definition 6.1 $x \wedge y \in p$ if and only if $x \in p$ and $y \in p$ and it follows that $(x \wedge y)^{\bullet} = x^{\bullet} \wedge y^{\bullet}$.
2. $(\neg x)^{\bullet} = \{p \in |points(\mathsf{B})| \mid \neg x \in p\}$. We use Proposition 6.10.
3. $(\text{и}a.x)^{\bullet} = \{p \in |points(\mathsf{B})| \mid \text{и}a.x \in p\}$. Suppose $p \in (\text{и}a.x)^{\bullet}$. Choose fresh a' (so $a'\#x, x^{\bullet}, p$). By (**Alpha**) $\text{и}a.x = \text{и}a'.(a'\ a){\cdot}x$. By definition $\text{и}a'.(a'\ a){\cdot}x \in p$. By Lemma 6.12 $(a'\ a){\cdot}x \in p$. By definition $p \in (a'\ a){\cdot}x^{\bullet}$. By Proposition 5.10 $p \in \mathsf{n}a'.(a'\ a){\cdot}x^{\bullet}$. By Lemma 5.7 $\mathsf{n}a'.(a'\ a){\cdot}x^{\bullet} = \mathsf{n}a.x^{\bullet}$.

Remark 6.24. Note the 'internal' and 'external' names in part 3 of the proof of Proposition 6.23. We begin with a 'internally restricted' with и. We use (**Alpha**) to apply an 'external' renaming of a to 'externally' fresh a'. This is picked up by the definition of n-filter and Proposition 5.10 moves to the 'external' И.

In part 3 $a\#x$ implies $a\#x^{\bullet}$—but this does not matter; we choose a *fresh*.

Theorem 6.25. $\text{-}^{\bullet}$ *is injective, thus* B *is isomorphic to a subalgebra of* $\mathsf{B}^{\bullet}$.

Proof. Suppose $x \in |\mathsf{B}|$ and $y \in |\mathsf{B}|$ are distinct. Suppose without loss of generality that $x \not\leq y$, so that $x \wedge \neg y \neq \bot$. By Lemma 6.7 $(x \wedge \neg y)\uparrow$ is an n-filter. By Theorem 6.17 there exists a point q containing $(x \wedge \neg y)\uparrow$. Then $x \wedge \neg y \in q$, hence $q \in x^{\bullet}$ and $q \notin y^{\bullet}$. The result follows by Proposition 6.23.

7 Nominal Stone duality

We introduce *nominal topological spaces*. As expected, the carrier set and topology are nominal sets and open sets are finitely-supported (this can restrict—sometimes considerably; see Example 5.3—the available open sets). Thus we cannot take arbitrary unions of open sets, but only finitely-supported unions. Banonas correspond to nominal topological spaces with additional properties which we call nominal Stone spaces with и (Definition 7.4). To the new-binder и corresponds on the topological side the semantic n (Definition 5.2); elements correspond to clopen sets (sets that are both open and closed), so clopens must be closed under n. The notion of compactness must also be subtly tweaked to take into account the role of n. We conclude with a duality theorem.

Definition 7.1. A **nominal topological space** T is a pair $(|\mathsf{T}|, \mathcal{O}_{\mathsf{T}})$ of a **carrier** nominal set $|\mathsf{T}|$ and equivariant set of **open sets** $\mathcal{O}_{\mathsf{T}} \subseteq pow(|\mathsf{T}|)$ such that:

- $\emptyset \in \mathcal{O}_{\mathsf{T}}$ and $|\mathsf{T}| \in \mathcal{O}_{\mathsf{T}}$
- $U \in \mathcal{O}_{\mathsf{T}} \wedge V \in \mathcal{O}_{\mathsf{T}}$ implies $U \cap V \in \mathcal{O}_{\mathsf{T}}$.
- $\mathcal{U} \in pow(\mathcal{O}_{\mathsf{T}})$ implies $\bigcup \mathcal{U} \in \mathcal{O}_{\mathsf{T}}$; we call this a **finitely-supported union**.

Call equivariant $f \in |\mathsf{T}_1| \rightarrow |\mathsf{T}_2|$ **continuous** when $V \in \mathcal{O}_{\mathsf{T}_2}$ implies $f^{-1}(V) \in \mathcal{O}_{\mathsf{T}_1}$. Write nTop for the category of nominal topological spaces and continuous maps.

Definition 7.2. Call $\mathcal{U}$ **n-closed** when $\text{И}a.\forall U.(U \in \mathcal{U} \Rightarrow \mathsf{n}a.U \in \mathcal{U})$. Call $\mathcal{U} \in pow(\mathcal{O}_{\mathsf{T}})$ a **cover** when $\bigcup \mathcal{U} = |\mathsf{T}|$. If $\mathcal{U}$ is a cover and is n-closed then call $\mathcal{U}$ an **n-cover**. Call T **n-compact** when every n-cover has a finite subcover.

Lemma 7.3. *If $\mathcal{U} \in pow(\mathcal{O}_{\mathsf{T}})$ is finite then $\text{И}a.\forall U.U \in \mathcal{U} \Rightarrow a\#U$. As a corollary, T is n-compact when every n-cover has an n-closed finite subcover.*

Proof. For finite $\mathcal{U}$, $supp(\mathcal{U}) = \bigcup\{supp(U) \mid U \in \mathcal{U}\}$ [14, Theorem 2.29]. We use part 1 of Lemma 5.6.

Definition 7.4. Call U **closed** when $|\mathsf{T}| \setminus U \in \mathcal{O}_{\mathsf{T}}$, and **clopen** when U is open and closed. Call T **totally separated** when for every $x, y \in |\mathsf{T}|$ there is a clopen U with $x \in U$ and $y \notin U$.

Say T is a **nominal topological space with** и when $\mathsf{n}a.U \in \mathcal{O}_{\mathsf{T}}$ for every clopen U.[5] Write $\mathsf{nTop}_{\text{И}}$ for the full subcategory of nTop on T with и.

A **nominal Stone space with** и is a totally separated n-compact nominal topological space with и. Write $\mathsf{nStone}_{\text{И}}$ for the full subcategory of $\mathsf{nTop}_{\text{И}}$ on nominal Stone spaces with и.

Definition 7.5. Given B in BAnona define $F(\mathsf{B})$ in nTop by:

- $|F(\mathsf{B})| = points(\mathsf{B})$ (Definition 6.18).
- $\mathcal{O}_{F(\mathsf{B})}$ is the closure of $\{x^\bullet \mid x \in |\mathsf{B}|\}$ (Definition 6.22) under finitely-supported unions. So $U \in \mathcal{O}_{F(\mathsf{B})}$ when $\exists M \in pow(|\mathsf{B}|).U = \bigcup\{x^\bullet \mid x \in M\}$.

Given $f : \mathsf{B} \to \mathsf{B}'$ in BAnona define $F(f) : F(\mathsf{B}') \to F(\mathsf{B})$ by $F(f)(p) = f^{-1}(p)$.

$F(\mathsf{B})$ is indeed a nominal topological space, for if $\mathcal{U} \in pow(\mathcal{O}_{F(\mathsf{B})})$ then $\bigcup \mathcal{U} = \bigcup\{x^\bullet \mid \exists U {\in} \mathcal{U}.x^\bullet {\subseteq} U\}$ is open. The next lemma shows that $F(f)$ is well-defined.

Lemma 7.6. *For $f : \mathsf{B} \to \mathsf{B}'$ in BAnona and $p \in points(\mathsf{B}')$, $f^{-1}(p) \in points(\mathsf{B})$.*

Proof. It is not hard to use the homomorphism properties of f and Theorem 4.2 to show that $f^{-1}(p)$ is an n-filter. Using Proposition 6.10 $\neg x \in f^{-1}(p)$ if and only if $x \notin f^{-1}(p)$, and it follows that $f^{-1}(p)$ is also maximal.

Lemma 7.7. *If $\mathcal{U}$ is n-closed then so is $\mathcal{V} = \{x^\bullet \mid \exists U {\in} \mathcal{U}.x^\bullet \subseteq U\}$.*

Proof. Choose b fresh (so $b\#\mathcal{U}, \mathcal{V}$). If $x^\bullet \in \mathcal{V}$ then $x^\bullet \subseteq U \in \mathcal{U}$. By Lemma 5.6 $\mathsf{n}b.(x^\bullet) \subseteq \mathsf{n}b.U$ and by assumption $\mathsf{n}b.U \in \mathcal{U}$.

Proposition 7.8. *$F(\mathsf{B})$ is totally separated and n-compact.*

Proof. Consider distinct $p, q \in |F(\mathsf{B})|$. Without loss of generality take $x \in p \setminus q$. By Definition 6.20, $p \in x^\bullet$ and $q \notin x^\bullet$. So $x^\bullet$ is an open set separating p and q. By Proposition 6.10 $points(\mathsf{B}) \setminus x^\bullet = (\neg x)^\bullet$, so $x^\bullet$ is also closed.

Consider an n-cover $\mathcal{U} \in pow(\mathcal{O}_{F(\mathsf{B})})$. It suffices to find a finite subcover of $\mathcal{V} = \{x^\bullet \mid \exists U {\in} \mathcal{U}.x^\bullet \subseteq U\}$, since for every $x^\bullet \in \mathcal{V}$ there exists $x^\bullet \subseteq U \in \mathcal{U}$.

[5] We need not require $\mathsf{n}a.U$ to be open for all U; of course, we do not *forbid* it either.

Write $X = \bigwedge_{fin}\{x' \mid \exists x.\neg x{\le}x' \wedge x^\bullet{\in}\mathcal{V}\}$ where $\bigwedge_{fin}$ denotes closure under finite intersections. So $\mathcal{V}$ has a finite subcover if and only if $\bot \in X$. By Proposition 6.23 and Lemma 7.7 X satisfies conditions 2 and 3 of Definition 6.1. If X satisfied also $\bot \notin X$ then X would be an n-filter; by Theorem 6.17 $X \subseteq p$ for some point p; it would follow that $p \notin \bigcup\mathcal{V}$, a contradiction. Therefore $\bot \in X$.

Lemma 7.9. *If $U \in F(\mathsf{B})$ is clopen then $U = x^\bullet$ for some $x \in |\mathsf{B}|$.*

Proof. By assumption $U = \bigcup\{x^\bullet \mid x^\bullet \subseteq U\}$ and $|F(\mathsf{B})| \setminus U = \bigcup\{x^\bullet \mid x^\bullet \subseteq |F(\mathsf{B})| \setminus U\}$. It follows that $\{x^\bullet \mid x^\bullet \subseteq U \vee x^\bullet \cap U = \varnothing\}$ covers $F(\mathsf{B})$. This cover is also n-closed, by part 1 and the corollary in Lemma 5.6. So it has a finite subcover by Proposition 7.8. The result follows by Proposition 6.23.

Proposition 7.10. *F is a functor from BAnona to $\mathsf{nStone}_{\text{И}}^{op}$.*

Proof. If $U{\in}\mathcal{O}_{\mathsf{F(B)}}$ is clopen then $U{=}x^\bullet$ for some $x{\in}|\mathsf{B}|$ by Lemma 7.9. By part 3 of Proposition 6.23 $\mathsf{n}a.(x^\bullet){=}(\text{И}a.x)^\bullet{\in}\mathcal{O}_{\mathsf{F(B)}}$. By Proposition 7.8 $F(\mathsf{B})$ is a nominal Stone space with И.

Consider $f : \mathsf{B} \to \mathsf{B}'$ in BAnona. By Lemma 7.6 $F(f)$ maps $|F(\mathsf{B}')|$ to $|F(\mathsf{B})|$. Continuity of $F(f)$ follows using the fact that $F(f)^{\text{-}1}(x^\bullet) = (f(x))^\bullet$.

Definition 7.11. Map $\mathsf{T} \in \mathsf{nStone}_{\text{И}}$ to a $G(\mathsf{T}) \in \mathsf{BAnona}$ defined by:

- $|G(\mathsf{T})| = \{U \in \mathcal{O}_{\mathsf{T}} \mid U \text{ is clopen}\}$.
- $\wedge$, $\neg$, and И are interpreted as intersection, complement, and n.

Given $f : \mathsf{T} \to \mathsf{T}'$ in $\mathsf{nStone}_{\text{И}}$ define $G(f) : G(\mathsf{T}') \to G(\mathsf{T})$ by $G(f)(U) = f^{\text{-}1}(U)$.

Lemma 7.12. *G is a functor from $\mathsf{nStone}_{\text{И}}^{op}$ to BAnona.*

Proof. If U is clopen then $\mathsf{n}a.U$ is open by Definition 7.4. $\mathsf{n}a.U$ is closed from the fact that $\mathsf{n}a.(|\mathsf{T}| \setminus U) = |\mathsf{T}| \setminus \mathsf{n}a.U$, immediate from part 3 of Lemma 5.6. It is routine to check that $G(\mathsf{T})$ is an object of BAnona.

Consider continuous $f : \mathsf{T} \to \mathsf{T}'$. If U is clopen then so is $f^{\text{-}1}(U)$. $G(f)$ is equivariant because f is, and preserves intersections and complements. Given $U \in \mathcal{O}_{\mathsf{T}'}$ a clopen, $G(f)(\mathsf{n}a.U) = \mathsf{n}a.G(f)(U)$ follows from Definitions 5.2 and 7.11 and the equivariance of f. Thus $G(f)$ is a morphism in BAnona.

Lemma 7.13. *Suppose $\mathsf{T} \in \mathsf{nTop}_{\text{И}}$ is n-compact and $\mathcal{U}$ is a finitely-supported set of closed sets with the* **finite intersection property***: the intersection of finitely many sets in $\mathcal{U}$ is nonempty. Then $\text{И}b.\forall U.U{\in}\mathcal{U} \Rightarrow \mathsf{n}b.U{\in}\mathcal{U}$ implies $\bigcap\mathcal{U} \neq \emptyset$.*

Theorem 7.14. *G defines an equivalence between BAnona and $\mathsf{nStone}_{\text{И}}^{op}$.*

Proof. We use [23, Theorem 1, Chapter IV, Section 4].
G is essentially surjective on objects. Given B in BAnona and $x \in |\mathsf{B}|$, $x^\bullet \in \mathcal{O}_{F(\mathsf{B})}$ is clopen. By Lemma 7.9 if $U \in \mathcal{O}_{F(\mathsf{B})}$ is clopen then $U = x^\bullet$ for some $x \in |\mathsf{B}|$. By Theorem 6.25, the map $\text{-}^\bullet$ defines an isomorphism between $GF(\mathsf{B})$ and B.
G is faithful. From the fact that nominal Stone spaces are totally separated.

G is full. Given T, T' in $\mathsf{nStone}_{\text{И}}$ and $u : G(\mathsf{T}') \to G(\mathsf{T})$ in BAnona we construct a morphism $v : \mathsf{T} \to \mathsf{T}'$ in $\mathsf{nStone}_{\text{И}}$, such that $G(v) = u$.

Define $\alpha_{\mathsf{T}} : \mathsf{T} \to FG(\mathsf{T})$ by $t \mapsto \{U \in G(\mathsf{T}) \mid t \in U\}$.

α_{T} is well defined: $\alpha_{\mathsf{T}}(t)$ is supported by $supp(t)$, and is indeed a maximal n-filter. We must also show that $a\#t$ and $U \in \alpha(t)$ imply $\mathsf{n}a.U \in \alpha(t)$; this follows from Proposition 5.10.

That α_{T} is injective follows, as in for the classical Stone duality, from the totally separatedness of the spaces. For surjectivity, consider a maximal n-filter $\mathcal{U} \in FG(\mathsf{T})$. This is a finitely-supported set of closed sets of T with the finite intersection property such that $U \in \mathcal{U}$ and $b\#\mathcal{U}$ imply $\mathsf{n}b.U \in \mathcal{U}$. By Lemma 7.13 there exists some $t \in \bigcap\mathcal{U} \subseteq |\mathsf{T}|$. That $\mathcal{U} = \alpha(t)$ follows from maximality of $\mathcal{U}$. α_{T} and α_{T}^{-1} are continuous. The proof is analoguous to the classical case, see [4].

We set $v = \alpha_{\mathsf{T}'}^{-1} \circ F(u) \circ \alpha_{\mathsf{T}}$. This is continuous and $G(v) = u$.

8 Conclusions

We have seen how Boolean algebras in nominal sets naturally support an operation corresponding to the И-quantifier from [18]. So nominal sets are sufficiently different that this paper is not a pure 'replay' of the standard proofs—on the contrary, the fine detail is extremely subtle.

In particular, as an empirical observation the proofs seem to 'want' и. They break for nominal Boolean algebras *without* и (remove и and its axioms from Definition 3.1, presumably to obtain a *Bano*). Without и we would need to use finitely-supported filters, rather than n-filters. But then the proofs of Lemma 6.13 and Theorem 6.17 would break. Banonas seem to be *natural*; a representation theorem for Banos is future work.

That our results are consistent with the *full* structure of classical logic (if we want it) is interesting, but note that our proofs do not depend on it. We believe that they should adapt to subreducts (e.g., Heyting algebras, distributive lattices) and expansions with operations/operators. A concrete example of a *Hanona* (Heyting algebras with и; weaken the initial three axioms appropriately and add $\text{и}a.(x{\Rightarrow}y) = (\text{и}a.x){\Rightarrow}(\text{и}a.y)$ in Figure 1) is a intuitionistic hybrid logic with $\downarrow$ (cf. Example 3.6 and [28]). We believe that, modulo an easy syntactic translation, the logic LG by Tiu [31] is another.

On related work, Sections 5.3 and 5.4 of [34] outline the Jónsson-Tarski representation of Boolean algebras with *operators*, i.e., *normal modalities*. Does it suffice to consider и and permutations as families of modalities? Yes, but this paper gives a strictly stronger result. Nominal sets have an 'external' theory of names and freshness given by permutations and support (extending the 'external' set intersection and complement corresponding to 'internal' conjunction and negation of Boolean algebra). An 'internal' notion of freshness is $x = \text{и}a.x$; a similar idea is used in cylindric/polyadic algebras and Lambda Abstraction Algebras [24]. Our challenge has been to make 'internal' and 'external' theories coincide for boolean connectives, *and* to represent и as n (Definition 5.2), permutation as permutation, and to satisfy e.g. $a\#x$ implies $x = \text{и}a.x$.

Some speculation on computational applications: fresh names create symmetry, since 'it does not matter' which are chosen. It is known by those working in satisfiability checkers (e.g. SMT solvers) that symmetry is important in reducing search space. Domain-specific symmetries are a major theme. Banonas and nominal sets give an explanatory foundation for *freshly generated resources*—a fairly common case. We might use this to extend problem input languages with explicit statements of these symmetries, thus improving and automating their recognition. Then this paper guarantees that a satisfiability checker based on nominal sets and n would be adequate for any classical logic with a $\mathsf{И}$.

On $pow(\mathbb{A})$, n is the unique function satisfying the axioms of Definition 3.1. We do not know whether n is uniquely determined by those axioms on arbitrary nominal powersets. (The map $X \mapsto \mathit{if}\ a\#X\ \mathit{then}\ X\ \mathit{else}\ \varnothing$ fails (**SelfDual**).)

Nominal powersets (Definition 5.1) have rich structure. In particular, function-symbols could be added to reflect *freshness* $X_{\#a} = \{x \in X \mid a\#x\}$ and *name-restriction* $\nu a.X = \{\pi{\cdot}x \mid x \in X,\ \pi \in \mathit{fix}(\mathit{supp}(X) \setminus \{a\})\}$ [16]. We do not believe that these can be represented using n. Note that n does not equal ν. For instance $X \subseteq \nu a.X$ is true, but $X \subseteq \mathsf{n}a.X$ is false in general.[6] So there are *two* natural name-restrictions on nominal sets. Nominal sets have been around for a decade, but their structure is not fully understood. For future work we hope to characterise the structure of nominal powersets, as complete atomic Boolean algebras do for 'ordinary' powersets.

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[6] The ν of [26], which Pitts calls *name-restriction*, is purely axiomatic and does not commit to any concrete representation.

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