Restart as a Computational Rule

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Classical Restart

The Natural Deduction **restart rule**:

$$\text{(Restart)} \qquad \frac{A}{B}$$

It was introduced by Dov Gabbay in 1984 in Goal Directed proof theory.
Michael Gabbay wrote it as a natural deduction rule in 2002.

Sorry? What's this? I can't hear… *'Side condition'?* Well if you insist…

Classical Restart

The side-condition is: a proof is valid if below every occurrence of restart from A to B, there is an occurrence of A.

For example $\dfrac{\top}{B}$ is not a valid proof. Here is a valid proof of Pierce's Law in Intuitionistic Logic (**IL**) with (Restart) (**ILR**):

$$
\dfrac{\dfrac{\dfrac{[A]}{B}\,\text{(Restart)}^{*}}{A \to B}\,(\to\text{I}) \qquad [(A \to B) \to A]}{\dfrac{A^{\dagger}}{((A \to B) \to A) \to A}\,(\to\text{I})}\,(\to\text{E})
$$

Deduction rules of ILR

$$(\rightarrow\mathrm{I})\ \dfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \rightarrow B} \qquad (\rightarrow\mathrm{E})\ \dfrac{A \quad A \rightarrow B}{B}$$

$$(\bot\mathrm{E})\ \dfrac{\bot}{B} \qquad (\mathrm{Restart})\ \begin{array}{c}\dfrac{A}{B}\\ \vdots \\ A\end{array}$$

ILR=CL

Theorem 1: Intuitionistic Logic plus (Restart) (ILR) has the same logical consequence as Classical Logic (CL).

Proof: One direction we have just seen, since IL+Pierce's Law has the same logical consequence as CL. Suppose now we have a proof involving restart. We may rewrite it as follows:

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{A \qquad [A \to B]}{B}\,(\to\mathrm{E})\\ \vdots \\ A\end{array}}{(A \to B) \to A}\,(\to\mathrm{I}) \qquad \dfrac{}{((A \to B) \to A) \to A}}{A}\,(\to\mathrm{E}).$$

Terms

Theorem 2: Intuitionistic Logic plus (Restart) inherits proof-normalisation from IL.

Proof: Later.

So we have a Natural Deduction system for Classical Logic. What about Curry-Howard?

Fix **variables** $x, y, z, \ldots$ and **covariables** $a, b, c, \ldots$. Terms are:

$$t ::= x \mid tt \mid \lambda x_A.t \mid \mathtt{R}[a_{A\supset B}]t \mid \mathtt{H}a_A.\,t.$$

We may drop type annotations. The covariable a is free in $\mathtt{R}[a]t$ and bound in $\mathtt{H}a.\,t$.

Typing rules

Contexts Γ are sets $x_1 : A_1, \ldots, x_k : A_k$ with x_i distinct. **Cocontexts** Δ are sets $a_1 : A_1, \ldots, a_n : A_n$, for a_i distinct. Typing rules include:

$$(\to\mathrm{I})\ \frac{\Gamma, x : A \vdash t : B \,;\, \Delta}{\Gamma \vdash (\lambda x_{A\to B}.t) : A \to B \,;\, \Delta}$$

$$(\to\mathrm{E})\ \frac{\Gamma \vdash t : A \to B \,;\, \Delta \quad \Gamma \vdash t' : A \,;\, \Delta}{\Gamma \vdash tt' : B \,;\, \Delta}$$

$$(\mathrm{Raise})\ \frac{\Gamma \vdash t : A \,;\, a : A, \Delta}{\Gamma \vdash (\mathtt{R}[a_{A\supset B}]t) : B \,;\, a : A, \Delta}$$

$$(\mathrm{Handle})\ \frac{\Gamma \vdash t : A \,;\, a : A, \Delta}{\Gamma \vdash (\mathtt{H}a_A.t) : A \,;\, \Delta}$$

ILR Structurally

Enrich judgements with **cocontexts** Δ to $\Gamma \vdash A ; \Delta$. Deduction rules include:

$$\dfrac{\begin{array}{c}[A] \\ \vdots \\ B ; \Delta\end{array}}{A \to B ; \Delta}\,(\to\text{I}) \qquad \dfrac{A \to B ; \Delta \quad A ; \Delta}{B ; \Delta}\,(\to\text{E})$$

$$\dfrac{A ; \Delta}{B ; A, \Delta}\,(\text{Raise}) \qquad \dfrac{A ; A, \Delta}{A ; \Delta}\,(\text{Handle})$$

Valid proofs of IL+(Restart) correspond to proofs with empty cocontext $\Delta = \emptyset$.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{[x : A]}{(\mathtt{R}[a_{A \supset B}]x) : B}\,(\text{Restart})^{a:A}
    }{
      \lambda x.\mathtt{R}[a]x : A \rightarrow B
    }\,(\rightarrow\text{I})
    \qquad
    [f : (A \rightarrow B) \rightarrow A]
  }{
    f(\lambda x.\mathtt{R}[a]x) : A
  }
}{
  \dfrac{}{\mathtt{H}a.\ f(\lambda x.\mathtt{R}[a]x) : A}\,(\text{Handle})^{a:A}
}
$$

$$
\dfrac{\mathtt{H}a.\ f(\lambda x.\mathtt{R}[a]x) : A}{\lambda f.\mathtt{H}a.\ f\big(\lambda x.\mathtt{R}[a_{A \supset B}]x\big) : ((A \rightarrow B) \rightarrow A) \rightarrow A}\,(\rightarrow\text{I})
$$

Some terms (II of II)

Add $\vee$ and $\perp$ to the typing system:

$$\frac{\Gamma \vdash t : \perp \,;\, \Delta}{\Gamma \vdash \mathtt{C}_A t : A \,;\, \Delta}\ (\perp\mathrm{E})$$

Here are terms of type $\neg\neg A \to A$ and $A \vee \neg A$:

$$\lambda f_{\neg\neg A}.\mathtt{H}a.\,\mathtt{C}_A f(\lambda x_A.\mathtt{R}[a_{A\supset\perp}]x)$$

$$\mathbf{Inr}_{A\vee\neg A}\lambda x_A.\mathtt{R}[a_{A\vee\neg A\supset\perp}]\mathbf{Inl}_{A\vee\neg A}x$$

Proof-Normalisation

Proof: Since (Restart) introduces no new constructors and generates no formulae, essential cases are as before. For example:

$$\dfrac{\dfrac{\begin{array}{c}[A]^1\\ \vdots \\ B\end{array}}{A \rightarrow B}\,(\rightarrow\mathsf{I})^1 \qquad \dfrac{\Pi}{A}}{B}\,(\rightarrow\mathsf{E}) \quad\Longrightarrow\quad \begin{array}{c}\dfrac{\Pi}{A}\\ \vdots \\ B\end{array}$$

Commutation cases remain, we consider just (→E) and (Handle):

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{A \to B}{C \mathbin{;} A \to B}\,\text{(R)}\\ \vdots \\ B \mathbin{;} A \to B\end{array}}{A \to B \mathbin{;} A \to B}\,\text{(→I)}^{1}}{A \to B}\,\text{(H)} \qquad \dfrac{\Pi}{A}}{B}\,\text{(→E)} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{A \to B \quad \dfrac{\Pi}{A}}{B}\,\text{(→E)}}{C \mathbin{;} B}\,\text{(R)}\\ \vdots \\ B \mathbin{;} B\end{array}}{A \to B \mathbin{;} B}\,\text{(→I)}^{1} \qquad \dfrac{\Pi \mathbin{;} B}{A \mathbin{;} B \mathbin{;} B}}{B \mathbin{;} B}\,\text{(→E)}}{B}\,\text{(H)}$$

Now each of the problematic instances of (Restart) is justified one line further down, by B rather than $A \to B$, and we can proceed with elimination of the essential case.

Reductions

$$(\lambda x.t)t' \rightsquigarrow t[x \mapsto t'] \qquad (\mathtt{H}a.\, t)t' \rightsquigarrow \mathtt{H}a.\, (t[\mathtt{R}[a]s \mapsto (\mathtt{R}[a]s)t'])$$

$$\mathtt{H}a.\, \mathtt{R}[a]t \rightsquigarrow t \quad (a \notin t)$$

Elimination rules get '**teleported**' into (multiple positions in) the term.

Compare with $\lambda\mu$-calculus. Arguably the calculus above improves on $\lambda\mu$, which (simplifying) has (Handle) only at $\perp$.

Other restarts: linear restart

$$\dfrac{\dfrac{\dfrac{A}{B}\,(\text{LRestart1})}{\vdots}\quad C \qquad \dfrac{\dfrac{B}{A}\,(\text{LRestart2})}{\vdots}\quad C}{C}\,(\text{LHandle})$$

Corresponds to $(A \to B) \vee (B \to A)$ and to linear Kripke models.

Normalising. ‘Teleportation’ sideways across the branches.

Other restarts: n-depth restart

$$
\begin{array}{c}
\dfrac{A_1 \ldots A_2}{B} \\
\vdots \\
A_1 \\
\vdots \\
A_2 \\
\vdots \\
\\
\vdots \\
A_n
\end{array}
$$

where no assumption in the proof of A_{i+1} may be discharged before A_i has been handled in the bottom part of the proof.

Corresponds to Rather Ghastly Axioms, and to models of maximal depth n. Normalising. 'Teleportation' as in Classical Restart, but raises occur in groups of n.

Conclusions

Restart is a methodology for strengthening logics. It does not do as much damage to normalisation as other methods (invention from scratch, adding constants). Because of its structural nature it is, at least as far as we can see, parametric over the underlying logic: we used nothing particular about IL.

Future work: More restart rules and intermediate logics, different base logics, examine the calculi generated. [Increasingly hysterically] Huge source of free normalising term calculi!!!!

Generalise restart from a structural modification of a logic (considered as a tree) to one of a general graph (automata). What is that co-context. There are things going on which could be generalised and which we do not understand.

Rather Ghastly Axioms

A characteristic axiom scheme for models of depth 1 is $((A \to B) \to A) \to A$. Call these $P^1_{A,B}$. A characteristic axiom scheme for depth $i+1$ is

$$P^{i+1}_{A_1,\dots,A_{i+1}} \stackrel{\text{def}}{=} P^{i}_{A_1,P^i_{A_2,\dots,A_{i+1}}}.$$

For example

$P^2_{A,B,C} = ((A \to \Big(((B \to C) \to B) \to B\Big)) \to A) \to A$.