One-and-a-halfth-order Logic

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First-Order Logic (FOL)

Fix countably infinitely many variable symbols $a, b, c, \ldots$. Let terms be:

$$t ::= a$$

Formulae or predicates are:

$$\phi ::= \bot \quad | \quad \phi \supset \phi \quad | \quad \forall a.\phi \quad | \quad t \approx t'.$$

Write $\equiv$ for syntactic identity.

Derivation

A context Φ and cocontext Ψ are finite and possibly empty sets of formulae.

A judgement is a pair $\Phi \vdash \Psi$ of a context and cocontext.

Valid judgements are inductively defined by:

$$(Axiom)\ \frac{}{\phi,\ \Phi \vdash \Psi,\ \phi} \qquad (\bot L)\ \frac{}{\bot,\ \Phi \vdash \Psi}$$

$$(\supset R)\ \frac{\phi,\ \Phi \vdash \Psi,\ \psi}{\Phi \vdash \Psi,\ \phi \supset \psi} \qquad (\supset L)\ \frac{\Phi \vdash \Psi,\ \phi \quad \psi,\ \Phi \vdash \Psi}{\phi \supset \psi,\ \Phi \vdash \Psi}$$

$$(\forall R)\ \frac{\Phi \vdash \Psi,\ \psi}{\Phi \vdash \Psi,\ \forall a.\psi}\ a \text{ fresh for } \Phi, \Psi \qquad (\forall L)\ \frac{\phi[a \mapsto t],\ \Phi \vdash \Psi}{\forall a.\phi,\ \Phi \vdash \Psi}$$

Hang on a moment

What are ϕ and ψ?

They are meta-variables ranging over formulae.

What are t and a?

They are meta-variables ranging over terms and variable symbols.

What is $\phi[a \mapsto t]$?

It is a meta-level operation which is only well-defined once we have a real predicate, a real variable symbol, and a real term.

What is 'a fresh for Φ and Ψ'?

It is a meta-level condition which is only well-defined once we have a real context and cocontext.

Schema

Quite a lot of things happen in the meta-level in First-Order Logic (FOL).

For example the following sequent

$$\vdash\ \forall a.\, \forall b.\, \phi \Leftrightarrow \forall b.\, \forall a.\, \phi$$

is derivable for every value of the meta-variable ϕ:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \vdash \phi}\,(Axiom)}{\forall b.\, \phi \vdash \phi}\,(\forall L)}{\forall a.\, \forall b.\, \phi \vdash \phi}\,(\forall L)}{\forall a.\, \forall b.\, \phi \vdash \forall a.\, \phi}\,(\forall R)}{\forall a.\, \forall b.\, \phi \vdash \forall b.\, \forall a.\, \phi}\,(\forall R)$$

Schema

However, the fact that this happens for all ϕ cannot be expressed in FOL.

Here are some other nice theorems:

1. $t \approx t' \;\vdash\; \phi[a{\mapsto}t] \Leftrightarrow \phi[a{\mapsto}t']$.
2. If $a \notin fv(\phi)$ then $\vdash\; (\forall a.\, \phi) \Leftrightarrow \phi$.

Logic of higher orders

This is often taken as an argument for higher-order logic (HOL).

In HOL, propositions have a type o and $\forall_\sigma$ is a constant with type $(\sigma\to o)\to o$, write just $\forall$ or $\forall : (\sigma\to o)\to o$.

Then a derivation of

$$\vdash\ \forall\lambda f.\big(\forall\lambda a.\forall\lambda b.fab \Leftrightarrow \forall\lambda b.\forall\lambda a.fab\big)$$

expresses that

$$\vdash\ \forall a.\,\forall b.\,\phi \Leftrightarrow \forall b.\,\forall a.\,\phi$$

holds for all ϕ, in one derivable sequent.

Here f has function type. If $a : \sigma$ and $b : \tau$ then $f : \sigma\to\tau\to o$ and 'fab is ϕ'.

Similarly:

1. $t \approx t' \ \vdash \ \phi[a{\mapsto}t] \Leftrightarrow \phi[a{\mapsto}t']$ in one-and-a-halfth-order logic becomes
$$t \approx t' \ \vdash \ \forall\lambda f.\ \big(ft \Leftrightarrow ft'\big).$$
in HOL.

   Note the types: f has function type and if $t : \sigma$ then $f : \sigma{\to}o$ and $\forall : ((\sigma{\to}o){\to}o){\to}o$.

2. If $a \notin fv(\phi)$ then $\vdash \ \forall a.\ \phi \Leftrightarrow \phi$ in one-and-a-halfth-order logic is not expressible in HOL.

Schema

One-and-a-halfth order logic addresses these problems in a different way.

We can state and prove our ‘test examples’ as single, derivable sequents.

This is reminiscent of algebraisations of first-order logic (such as cylindric algebra).

Here goes

Sorts are defined by:

$$\sigma ::= \mathbb{F} \quad | \quad \mathbb{T} \quad | \quad [\mathbb{A}]\sigma$$

We call $\mathbb{F}$ formulae, we call $\mathbb{T}$ terms, and we call $\mathbb{A}$ atoms.

Fix atoms $a, b, c, \ldots$, and variables $T, U, X, Y, \ldots$. Atoms all have sort $\mathbb{A}$. Variables may have any sort, but we tend to let T and U have sort $\mathbb{T}$ (we call them term variables) and X and Y have sort $\mathbb{F}$ (we call them predicate variables).

Here goes

Permutations π are finitely-supported bijections on atoms. A bijection is finitely supported when $\pi(a) \neq a$ for some finite set of atoms a, but for all other atoms $\pi(b) = b$.

(So π is 'mostly' the identity.)

Terms are:

$$t ::= a_{\mathbb{A}} \quad | \quad (\pi \cdot X_\sigma)_\sigma \quad | \quad ([a_{\mathbb{A}}]t_\sigma)_{[\mathbb{A}]\sigma} \quad | \\ (\forall t_{[\mathbb{A}]\mathbb{F}})_{\mathbb{F}} \quad | \quad (t_{\mathbb{F}} \supset t_{\mathbb{F}})_{\mathbb{F}} \quad | \quad \bot_{\mathbb{F}} \quad | \quad (t_{\mathbb{T}} \approx t_{\mathbb{T}})_{\mathbb{F}} \quad | \\ sub(t_{[\mathbb{A}]\sigma}, s_{\mathbb{T}})_\sigma.$$

We tend to write $sub([a]t, s)$ as $t[a{\mapsto}s]$.

Some example terms

Write $\neg\phi$ for $\phi \supset \bot$, write $\phi \wedge \phi'$ for $\neg(\phi \supset \neg\phi')$, write $\phi \Leftrightarrow \phi'$ for $(\phi \supset \phi') \wedge (\phi' \supset \phi)$, write $\phi \vee \phi'$ for $(\neg\phi) \supset \phi'$, write $\top$ for $\bot \supset \bot$.

1. $\forall[a]\forall[b]X \Leftrightarrow \forall[b]\forall[a]X$.
2. $T \approx T'$.
3. $X[a{\mapsto}T] \Leftrightarrow X[a{\mapsto}T']$.
4. $\forall[a]X \Leftrightarrow X$.

Meaning of 'free variables of' with predicate unknowns X.

A freshness $F \equiv a\#t$ is a pair of an atom and a term.

$$\frac{}{a\#b}(\#ab) \qquad \frac{a\#t_1 \ \cdots \ a\#t_n}{a\#\mathsf{f}(t_1,\ldots,t_n)}(\#\mathsf{f}) \qquad \frac{}{a\#[a]t}(\#[]a)$$

$$\frac{a\#t}{a\#[b]t}(\#[]b) \qquad \frac{\pi^{-1}\cdot a\#X}{a\#\pi\cdot X}(\#X)$$

Let Δ be a set of freshnesses. Write $\Delta \vdash F$ when F follows from Δ (say Δ entails F).

Here f is semi-formal; $\mathsf{f} \in \{\forall, \supset, \bot, \approx, sub\}$.

Sequent derivation rules

$$\frac{}{\phi,\ \Phi\ \vdash_{\triangle}\ \Psi,\ \phi}\,(Axiom) \qquad \frac{}{\bot,\ \Phi\ \vdash_{\triangle}\ \Psi}\,(\bot L)$$

$$\frac{\Phi\ \vdash_{\triangle}\ \Psi,\ \phi \qquad \psi,\ \Phi\ \vdash_{\triangle}\ \Psi}{\phi \supset \psi,\ \Phi\ \vdash_{\triangle}\ \Psi}\,(\supset L) \qquad \frac{\phi,\ \Phi\ \vdash_{\triangle}\ \Psi,\ \psi}{\Phi\ \vdash_{\triangle}\ \Psi,\ \phi \supset \psi}\,(\supset R)$$

$$\frac{\phi',\ \Phi\ \vdash_{\triangle}\ \Psi \qquad \phi'\ \vdash_{\triangle}^{\mathsf{SUB}}\ \phi[a{\mapsto}t]}{\forall[a]\phi,\ \Phi\ \vdash_{\triangle}\ \Psi}\,(\forall L)$$

$$\frac{\Phi\ \vdash_{\triangle}\ \Psi,\ \psi \qquad \Delta\ \vdash\ a\#\Phi, \Psi}{\Phi\ \vdash_{\triangle}\ \Psi,\ \forall[a]\psi}\,(\forall R)$$

Em. . . just a few more sequent derivation rules

$$\frac{}{\Phi \vdash \Psi,\, t \approx t}\,(\approx R)$$

$$\frac{\phi',\, \Phi \vdash \Psi \qquad \phi' \vdash^{\mathsf{SUB}}_{\triangle} \phi''[a{\mapsto}t'] \qquad \phi \vdash^{\mathsf{SUB}}_{\triangle} \phi''[a{\mapsto}t]}{t'{\approx}t,\, \phi,\, \Phi \vdash_{\triangle} \Psi}\,(\approx L)$$

$$\frac{\phi',\, \Phi \vdash_{\triangle} \Psi \qquad \phi' \vdash^{\mathsf{SUB}}_{\triangle} \phi}{\phi,\, \Phi \vdash_{\triangle} \Psi}\,(StructL)$$

$$\frac{\Phi \vdash_{\triangle} \Psi,\, \psi' \qquad \psi' \vdash^{\mathsf{SUB}}_{\triangle} \psi}{\Phi \vdash_{\triangle} \Psi,\, \psi}\,(StructR)$$

We will discuss $\vdash^{\mathsf{SUB}}$ later.

Example derivations

$$\dfrac{\dfrac{\forall[a]\forall[b]X \vdash X \qquad a\#\forall[b]X}{\forall[a]\forall[b]X \vdash \forall[a]X}(\forall R) \qquad b\#\forall[a]\forall[b]X}{\forall[a]\forall[b]X \vdash \forall[b]\forall[a]X}(\forall R)$$

$$\dfrac{\dfrac{\dfrac{}{X \vdash X}(Axiom) \qquad X \vdash^{\mathsf{SUB}} X[b\mapsto b]}{\forall[b]X \vdash X}(\forall L) \qquad \forall[b]X \vdash^{\mathsf{SUB}} (\forall[b]X)[a\mapsto a]}{\forall[a]\forall[b]X \vdash X}(\forall L)$$

Semantics in FOL:

"For all ϕ and ψ, $\forall a.\, \forall b.\, \phi \vdash \forall b.\, \forall a.\, \psi$."

More of the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{b\#[b]X}\,(\#[]a)}{b\#\forall[b]X}\,(\#\mathsf{f})}{b\#[a]\forall[b]X}\,(\#[]a)}{b\#\forall[a]\forall[b]X}\,(\#\mathsf{f})$$

Another example derivation

$$\dfrac{\dfrac{}{X[a{\mapsto}T'] \vdash X[a{\mapsto}T']}(Axiom) \quad \begin{array}{c} X[a{\mapsto}a][a{\mapsto}T'] \vdash^{\mathsf{SUB}} X[a{\mapsto}T'], \\ X[a{\mapsto}a][a{\mapsto}T] \vdash^{\mathsf{SUB}} X[a{\mapsto}T] \end{array}}{T' \approx T,\ X[a{\mapsto}T] \vdash X[a{\mapsto}T']}(\approx L)$$

Semantics in FOL:

"For all t and t' and ϕ, $t' \approx t,\ \phi[a{\mapsto}t] \vdash \phi[a{\mapsto}t']$."

One more example derivation

$$\dfrac{\dfrac{}{X \vdash_{\triangle} X}(Axiom) \qquad a\#X \vdash a\#X}{X \vdash_{a\#X} \forall[a]X}(\forall R)$$

Semantics in FOL:

"For all ϕ and a, if $a \notin fv(\phi)$ then $\phi \vdash \forall a.\, \phi$."

A nice theorem:

$$\frac{\Phi \vdash_{\triangle} \Psi, \phi \qquad \phi, \Phi \vdash_{\triangle} \Psi}{\Phi \vdash_{\triangle} \Psi} (Cut)$$

Theorem (cut-elimination): Cut is eliminable.

The cut-elimination procedure is almost standard — but details of α-renaming form part of the derivation.

Meaning of $\vdash^{\mathsf{SUB}}$

Write $t \vdash^{\mathsf{SUB}}_{\Delta} u$ when $t = u$ is derivable from assumptions Δ using the following axioms:

$$
\begin{array}{rrcl}
(f{\mapsto}) & \mathsf{f}(u_1, \ldots, u_n)[a{\mapsto}t] & = & \mathsf{f}(u_1[a{\mapsto}t], \ldots, u_n[a{\mapsto}t]) \\
([b]{\mapsto}) & b\#t{\longrightarrow}([b]u)[a{\mapsto}t] & = & [b](u[a{\mapsto}t]) \\
(var{\mapsto}) & a[a{\mapsto}t] & = & t \\
(u{\mapsto}) & a\#u{\longrightarrow}u[a{\mapsto}t] & = & u \\
(ren{\mapsto}) & b\#u{\longrightarrow}u[a{\mapsto}b] & = & (b\ a)\cdot u \\
(perm) & a, b\#t{\longrightarrow}(a\ b)\cdot t & = & t
\end{array}
$$

Permutation action

$$\pi \cdot a \equiv \pi(a) \qquad \pi \cdot (\pi' \cdot X) \equiv (\pi \circ \pi') \cdot X$$

$$\pi \cdot [a]t \equiv [\pi(a)](\pi \cdot t)$$

$$\pi \cdot \mathsf{f}(t_1, \ldots, t_n) \equiv \mathsf{f}(\pi \cdot t_1, \ldots, \pi \cdot t_n)$$

Some more nice theorems:

Theorem: First-order logic corresponds in a natural and formal sense precisely to closed terms (terms mentioning no variables), like $\forall[a](a \approx a)$.

Theorem: Cylindric algebra corresponds in a natural and formal sense precisely to cylindric terms (terms possibly mentioning variables, but not mentioning substitution), like $a \approx b$ (corresponding to 'd_{ab}' in cylindric algebras) or $\neg\forall[a]\neg X$ ('$c_a X$').

Relation to HOL

Not direct since we can express $a\#t$ and HOL cannot.

Also, suppose $X : o$ and $t : \mathbb{T}$. Then $X[a{\mapsto}t]$ corresponds to ft in HOL where $f : \mathbb{T}{\to}o$. However, $X[a{\mapsto}t][a'{\mapsto}t']$ corresponds to $f'tt'$ where $f' : \mathbb{T}{\to}\mathbb{T}{\to}o$. Similarly $X[a{\mapsto}t][a'{\mapsto}t'][a''{\mapsto}t'']\ldots$

This is type raising.

In one-and-a-halfth-order logic, X remains at sort o throughout and the universal quantification implicit in the use of X allows arbitrary numbers of substitutions.

Relation to HOL

On the other hand, one-and-a-halfth-order logic is manifestly not (fully) higher-order. For example we can write

$$X \vdash Y$$

meaning in FOL

> "For all formulae ϕ and ψ, $\phi \vdash \psi$."

(A silly but perfectly well-formed judgement.)

In HOL we can write this as $\vdash \forall \phi, \psi.\ \phi \supset \psi$.

However we can also write $\vdash \forall \psi.\ \big((\forall \phi.\ \phi) \supset \psi\big)$.

This is not possible in one-and-a-halfth-order logic: the universal quantification is implicit, and top-level (like ML type quantifiers).
$\forall[X]X \vdash Y$ is not syntax.

The sequent system is equivalent to the following 'Hilbert-style' axiomatisation:

(Props) $P \supset Q \supset P \;=\; \top \qquad \neg\neg P \supset P \;=\; \top$

$$(P \supset Q) \supset (Q \supset R) \supset (P \supset R) \;=\; \top \qquad \bot \supset P \;=\; \top$$

(Quants) $\forall[a]\top = \top \qquad \forall[a]P \supset P[a{\mapsto}T] \;=\; \top$

$$\forall[a](P \wedge Q) \Leftrightarrow \forall[a]P \wedge \forall[a]Q \;=\; \top$$

$$a\#P \longrightarrow \forall[a](P \supset Q) \Leftrightarrow P \supset \forall[a]Q \;=\; \top$$

Write it just $\vdash_{\Delta}$.

$$\frac{}{t \vdash_{\Delta} t}\,(Axiom) \qquad \frac{t \vdash_{\Delta} u}{C[t] \vdash_{\Delta} C[u]}\,(Cong)$$

$$\frac{f(t_1[a\mapsto t'],\ldots,t_n[a\mapsto t']) \vdash_{\Delta} u}{f(t_1,\ldots,t_n)[a\mapsto t'] \vdash_{\Delta} u}\,(fL) \qquad \frac{t \vdash_{\Delta} f(u_1[a\mapsto u'],\ldots,t'_n[a\mapsto u'])}{t \vdash_{\Delta} f(u_1,\ldots,u_n)[a\mapsto u']}\,(fR)$$

$$\frac{[b](t[a\mapsto t']) \vdash_{\Delta} u \quad \Delta \vdash b\#t'}{([b]t)[a\mapsto t'] \vdash_{\Delta} u}\,(absL) \qquad \frac{t \vdash_{\Delta} [b](u[a\mapsto u']) \quad \Delta \vdash b\#u'}{t \vdash_{\Delta} ([b]u)[a\mapsto u']}\,(absR)$$

$$\frac{t \vdash_{\Delta} u \quad \Delta \vdash a,b\#t}{(a\,b)\cdot t \vdash_{\Delta} u}\,(varL) \qquad \frac{t \vdash_{\Delta} u \quad \Delta \vdash a,b\#u}{t \vdash_{\Delta} (a\,b)\cdot u}\,(varR)$$

$$\frac{t \vdash_{\Delta} u}{a[a{\mapsto}t] \vdash_{\Delta} u}\,(atmL) \qquad \frac{t \vdash_{\Delta} u}{t \vdash_{\Delta} a[a{\mapsto}u]}\,(atmR)$$

$$\frac{t \vdash_{\Delta} u \quad \Delta \vdash a\#t}{t[a{\mapsto}t'] \vdash_{\Delta} u}\,(\#L) \qquad \frac{t \vdash_{\Delta} u \quad \Delta \vdash a\#u}{t \vdash_{\Delta} u[a{\mapsto}u']}\,(\#R)$$

$$\frac{(b\,a) \cdot t \vdash_{\Delta} u \quad \Delta \vdash b\#t}{t[a{\mapsto}b] \vdash_{\Delta} u}\,(renL) \qquad \frac{t \vdash_{\Delta} (b\,a) \cdot u \quad \Delta \vdash b\#u}{t \vdash_{\Delta} u[a{\mapsto}b]}\,(renR)$$

Theorem: Cut is admissible for $\vdash_{\Delta}$.

$$\frac{t \vdash_{\Delta} u \quad u \vdash_{\Delta} v}{t \vdash_{\Delta} v}\,(Cut)$$

Conclusions

One-and-a-halfth-order logic enriches 'normal FOL' with predicate unknowns; thus enabling us to reason universally on predicates.

This is like the universal quantification implicit in a variable in a universal algebra judgement $t = u$. And indeed, one-and-a-halfth-order logic arose from an algebraisation of first-order logic.

Conclusions

Unlike what you might expect, we do not use a hierarchy of types to manage α-equivalence, λ-binding to handle free/bound variables, and function application to manage substitution.

Instead, we use abstraction $[a]X$, freshness $a\#X$, and an explicit axiomatisation of substitution.

The axiomatisation of substitution is syntax-directed and susceptible to syntax-directed proof-search (where the proof is of the equality of two terms).

Conclusions

We can reason about classes of predicates, using predicate variables.

We avoid the full power (and undecidability) of HOL, but seem to end up in something which is not a subset of that system.

α-equivalence is part of the derivation tree. Gives the logic extra detail, but also extra proof principles; i.e. we can reason about unknown predicates also under abstractors such as $\forall$ or λ, without incurring type-raising and thus function-spaces.

There is a close link to algebraisations of quantifier logics.

For further work, how about...

- Two-and-a-halfth-order logic (where you can abstract X)?
- Implementation and automation?
- Semantics (aside from in FOL)?