Term sequent logics: cut your terms

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Benefits of cut elimination

Formal:

– Consistency
– Non-derivability
– Completeness for derivation-search algorithms
– Decidability results for fragments
– Extensions

Philosophical:

– Syntactic characterisation of what it is to 'be a logic'
– Grounds a theory of meaning

Cut elimination and axioms

- These properties do not follow in the presence of axioms. We need axioms to derive:

$$\forall \mathtt{x} \forall \mathtt{y} (\mathtt{x} \times \mathtt{y} \approx \mathtt{y} \times \mathtt{x})$$

- Sequents operate on the sentential level, but many interesting formal systems operate on the level of terms. For example:
  - Arithmetic
  - Lambda calculus
- Q. How can we get the formal and philosophical benefits of cut elimination for 'term' logics?
  A. Use ***term sequents***.

A brief overview of term sequents

Term sequent logic has two judgement forms:

– a *truth judgement* $\Gamma \vdash \Delta$

– a *term judgement* $\Gamma \vdash \Theta :: \Theta'$, or $\Gamma \vdash \Theta :\!\!- \Theta'$

The symbol $:\!\!-$ indicates a relation (equality; reducibility; definedness;...) between Θ and Θ' just as $\vdash$ indicates an entailment relation between Γ and Δ.

- The structure of Θ and Θ' depends on the term-formers we want term sequent derivation rules for.
- The challenge is to find a suitable term sequent structure together with syntax-directed term sequent derivation rules correlating to the theory we wish to examine.

Three term sequent logics

- *Numbers with addition and multiplication.*
  - $\Gamma \vdash \Theta :: \Theta'$
  - Θ and Θ' are multisets of multisets. Intuitively they each are a sum of products.
- *λ-calculus.*
  - $\Gamma \vdash \Theta :\!- \mathtt{t}$
  - Θ is a tree, $\mathtt{t}$ is a term. Intuitively, Θ corresponds to an η-expanded head normal form.
- *Boolean algebra*
  - $\Gamma \vdash \Theta :\!- \Theta'$
  - Θ is a set.

Related work

- *Rewriting*
  - The truth-judgement form of rewriting is weak ('s rewrites to t')
  - Rewrites are not easily conditionalised.
- *Deduction Modulo*
  - Not syntax directed.
  - Again, conditionalising on rewrites is not simple.

Basic syntax of arithmetic

- Define ***terms*** $\mathtt{t}$ and ***sentences*** $\mathtt{A}$ by:

$$\begin{array}{lll} \mathtt{t} & ::= & \mathtt{x}, \mathtt{y}, \mathtt{z}, \ldots \mid (\mathtt{t}_1 + \mathtt{t}_2) \mid (\mathtt{t}_1 \times \mathtt{t}_2) \mid \mathtt{0} \mid \mathtt{1} \\ \mathtt{A} & ::= & \mathtt{t}_1 \approx \mathtt{t}_2 \mid \mathtt{A}_1 \wedge \mathtt{A}_2 \mid \neg \mathtt{A} \mid \forall \mathtt{x} \mathtt{A} \end{array}$$

- $\overline{\mathtt{t}}, \overline{\mathtt{s}} \ldots$ range over finite multisets of terms.
  $\Theta, \Theta', \Psi, \Psi'$ range over finite multisets of finite multisets of terms.
  $()$ is the empty multiset.

- A ***term sequent*** is a tuple $\Gamma \vdash \Theta :: \Theta'$.
  A ***sentence sequent*** is a pair $\Gamma \vdash \Delta$.

Term sequent rules

$$\frac{}{\Gamma \vdash \Theta :: \Theta}\ (Ax_{::}) \quad \Theta \text{ atomic} \qquad \frac{\Gamma \vdash \Theta, () ::}{\Gamma \vdash \Delta}\ (T_\perp)$$

$$\frac{\Gamma \vdash \overline{t}t_1, \Theta_1 :: \overline{t}t_2, \Theta_2}{\Gamma, t_i \approx t_j \vdash \Theta_1 :: \Theta_2}\ (\approx L) \quad \begin{array}{l} i{=}1, j{=}2 \\ \text{or} \\ i{=}2, j{=}1 \end{array} \qquad \frac{\Gamma \vdash t_1 :: t_2}{\Gamma \vdash t_i \approx t_j, \Delta}\ (\approx R) \quad \begin{array}{l} i{=}1, j{=}2 \\ \text{or} \\ i{=}2, j{=}1 \end{array}$$

$$\frac{\Gamma \vdash \Theta, \overline{t} :: \Theta'}{\Gamma \vdash \Theta, 1\overline{t} :: \Theta'}\ (1L) \quad \frac{\Gamma \vdash \Theta' :: \Theta, \overline{t}}{\Gamma \vdash \Theta' :: \Theta, 1\overline{t}}\ (1R) \quad \frac{\Gamma \vdash \Theta :: \Theta'}{\Gamma \vdash \Theta, 0\overline{t} :: \Theta'}\ (0L) \quad \frac{\Gamma \vdash \Theta' :: \Theta}{\Gamma \vdash \Theta' :: \Theta, 0\overline{t}}\ (0R)$$

$$\frac{\Gamma \vdash \Theta, \overline{t}t_1, \overline{t}t_2 :: \Theta'}{\Gamma \vdash \Theta, \overline{t}(t_1{+}t_2) :: \Theta'}\ (+L) \qquad \frac{\Gamma \vdash \Theta' :: \Theta, \overline{t}t_1, \overline{t}t_2}{\Gamma \vdash \Theta' :: \Theta, \overline{t}(t_1{+}t_2)}\ (+R)$$

$$\frac{\Gamma \vdash \Theta, \overline{t}t_1t_2 :: \Theta'}{\Gamma \vdash \Theta, \overline{t}(t_1{\times}t_2) :: \Theta'}\ (\times L) \qquad \frac{\Gamma \vdash \Theta' :: \Theta, \overline{t}t_1t_2}{\Gamma \vdash \Theta' :: \Theta, \overline{t}(t_1{\times}t_2)}\ (\times R)$$

Sentence sequent rules

$$\frac{\Gamma, \mathtt{A}_i \vdash \Delta}{\Gamma, \mathtt{A}_1 \wedge \mathtt{A}_2 \vdash \Delta}\ (\wedge L) \quad 1 \leqslant i \leqslant 2 \qquad \frac{\Gamma \vdash \mathtt{A}_1, \Delta \quad \Gamma \vdash \mathtt{A}_2, \Delta}{\Gamma \vdash \mathtt{A}_1 \wedge \mathtt{A}_2, \Delta}\ (\wedge R) \qquad \frac{\Gamma \vdash \mathtt{C}, \Delta \quad \Gamma, \mathtt{C} \vdash \Delta}{\Gamma \vdash \Delta}\ (Cut)$$

$$\frac{\Gamma, \mathtt{A}[\mathtt{x}/\mathtt{t}] \vdash \Delta}{\Gamma, \forall \mathtt{x}\mathtt{A} \vdash \Delta}\ (\forall L) \qquad \frac{\Gamma \vdash \mathtt{A}, \Delta}{\Gamma \vdash \forall \mathtt{x}\mathtt{A}, \Delta}\ (\forall R) \quad \mathtt{x} \text{ not free in } \Gamma, \Delta \qquad \frac{\Gamma, \mathtt{A} \vdash \Delta}{\Gamma \vdash \neg \mathtt{A}, \Delta}\ (\neg R) \qquad \frac{\Gamma \vdash \mathtt{A}, \Delta}{\Gamma, \neg \mathtt{A} \vdash \Delta}\ (\neg L)$$

Conspicuously absent are...

- Axiom rule:

$$\frac{}{\Gamma, \mathtt{A} \vdash \mathtt{A}, \Delta}$$

- Equality rule:

$$\frac{\Gamma \vdash \mathtt{A}[\mathtt{x}/\mathtt{t}],\ \Delta}{\Gamma,\ \mathtt{t} \approx \mathtt{t}' \vdash \mathtt{A}[\mathtt{x}/\mathtt{t}'],\ \Delta}$$

Both are *derivable* in term sequent logic.

Term sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\ ::}\,(Ax_{::})}{\vdash \mathtt{x0} ::}\,(\mathtt{0}L)}{\vdash \mathtt{x0} :: \mathtt{0}}\,(\mathtt{0}R)}{\vdash \mathtt{x}\times\mathtt{0} :: \mathtt{0}}\,(\times L)}{\vdash \mathtt{x}\times\mathtt{0} \approx \mathtt{0}}\,(\approx R)$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \mathtt{xy},\mathtt{x}'\mathtt{y},\mathtt{x}'\mathtt{y}' :: \mathtt{x}'\mathtt{y},\mathtt{x}'\mathtt{y}',\mathtt{xy}}\,Ax_{::}}{\Gamma \vdash \mathtt{xy},\mathtt{x}'\mathtt{y},\mathtt{x}'\times\mathtt{y}' :: \mathtt{x}'\mathtt{y},\mathtt{x}'\mathtt{y}',\mathtt{x}\times\mathtt{y}}\,(\times L),(\times R)}{\Gamma,\mathtt{x}\approx\mathtt{x}' \vdash \mathtt{x}'\mathtt{y},\mathtt{x}'\times\mathtt{y}' :: \mathtt{x}'\mathtt{y}',\mathtt{x}\times\mathtt{y}}\,(\approx L)}{\Gamma,\mathtt{x}\approx\mathtt{x}',\mathtt{y}\approx\mathtt{y}' \vdash \mathtt{x}'\times\mathtt{y}' :: \mathtt{x}\times\mathtt{y}}\,(\approx L)}{\Gamma,\mathtt{x}\approx\mathtt{x}',\mathtt{y}\approx\mathtt{y}' \vdash \mathtt{x}'\times\mathtt{y}' \approx \mathtt{x}\times\mathtt{y},\Delta}\,(\approx R)$$

Term sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\ ::}\,(Ax_{::})}{\vdash \mathtt{x0} ::}\,(\mathtt{0}L)}{\vdash \mathtt{x0} :: \mathtt{0}}\,(\mathtt{0}R)}{\vdash \mathtt{x}\times\mathtt{0} :: \mathtt{0}}\,(\times L)}{\vdash \mathtt{x}\times\mathtt{0} \approx \mathtt{0}}\,(\approx R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z} :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(Ax_{::})}{\vdash \mathtt{x},(\mathtt{y}+\mathtt{z}) :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(+L)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(+L)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: (\mathtt{x}+\mathtt{y}),\mathtt{z}}\,(+R)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: (\mathtt{x}+\mathtt{y})+\mathtt{z}}\,(+R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \mathtt{xy},\mathtt{x}'\mathtt{y},\mathtt{x}'\mathtt{y}' :: \mathtt{x}'\mathtt{y},\mathtt{x}'\mathtt{y}',\mathtt{xy}}\,Ax_{::}}{\Gamma \vdash \mathtt{xy},\mathtt{x}'\mathtt{y},\mathtt{x}'\times\mathtt{y}' :: \mathtt{x}'\mathtt{y},\mathtt{x}'\mathtt{y}',\mathtt{x}\times\mathtt{y}}\,(\times L),(\times R)}{\Gamma,\mathtt{x}\approx\mathtt{x}' \vdash \mathtt{x}'\mathtt{y},\mathtt{x}'\times\mathtt{y}' :: \mathtt{x}'\mathtt{y}',\mathtt{x}\times\mathtt{y}}\,(\approx L)}{\Gamma,\mathtt{x}\approx\mathtt{x}',\mathtt{y}\approx\mathtt{y}' \vdash \mathtt{x}'\times\mathtt{y}' :: \mathtt{x}\times\mathtt{y}}\,(\approx L)}{\Gamma,\mathtt{x}\approx\mathtt{x}',\mathtt{y}\approx\mathtt{y}' \vdash \mathtt{x}'\times\mathtt{y}' \approx \mathtt{x}\times\mathtt{y},\Delta}\,(\approx R)$$

Term sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\ ::}\,(Ax_{::})}{\vdash \mathtt{x0} ::}\,(\mathtt{0}L)}{\vdash \mathtt{x0} :: \mathtt{0}}\,(\mathtt{0}R)}{\vdash \mathtt{x}\times\mathtt{0} :: \mathtt{0}}\,(\times L)}{\vdash \mathtt{x}\times\mathtt{0} \approx \mathtt{0}}\,(\approx R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z} :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(Ax_{::})}{\vdash \mathtt{x},(\mathtt{y}+\mathtt{z}) :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(+L)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(+L)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: (\mathtt{x}+\mathtt{y}),\mathtt{z}}\,(+R)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: (\mathtt{x}+\mathtt{y})+\mathtt{z}}\,(+R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash \mathtt{xy},\mathtt{x'y},\mathtt{x'y'} :: \mathtt{x'y},\mathtt{x'y'},\mathtt{xy}}{\Gamma \vdash \mathtt{xy},\mathtt{x'y},\mathtt{x'}\times\mathtt{y'} :: \mathtt{x'y},\mathtt{x'y'},\mathtt{x}\times\mathtt{y}}\,(\times L),(\times R)}{\Gamma,\mathtt{x}\approx\mathtt{x'} \vdash \mathtt{x'y},\mathtt{x'}\times\mathtt{y'} :: \mathtt{x'y'},\mathtt{x}\times\mathtt{y}}\,(\approx L)}{\Gamma,\mathtt{x}\approx\mathtt{x'},\mathtt{y}\approx\mathtt{y'} \vdash \mathtt{x'}\times\mathtt{y'} :: \mathtt{x}\times\mathtt{y}}\,(\approx L)}{\Gamma,\mathtt{x}\approx\mathtt{x'},\mathtt{y}\approx\mathtt{y'} \vdash \mathtt{x'}\times\mathtt{y'} \approx \mathtt{x}\times\mathtt{y},\Delta}\,(\approx R)$$

Term sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\ ::}\,(Ax_{::})}{\vdash \mathtt{x0} ::}\,(\mathtt{0}L)}{\vdash \mathtt{x0} :: \mathtt{0}}\,(\mathtt{0}R)}{\vdash \mathtt{x}\times\mathtt{0} :: \mathtt{0}}\,(\times L)}{\vdash \mathtt{x}\times\mathtt{0} \approx \mathtt{0}}\,(\approx R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z} :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(Ax_{::})}{\vdash \mathtt{x},(\mathtt{y}+\mathtt{z}) :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(+L)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(+L)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: (\mathtt{x}+\mathtt{y}),\mathtt{z}}\,(+R)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: (\mathtt{x}+\mathtt{y})+\mathtt{z}}\,(+R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \mathtt{xy},\mathtt{x'y},\mathtt{x'y'} :: \mathtt{x'y},\mathtt{x'y'},\mathtt{xy}}}{\Gamma \vdash \mathtt{xy},\mathtt{x'y},\mathtt{x'}\times\mathtt{y'} :: \mathtt{x'y},\mathtt{x'y'},\mathtt{x}\times\mathtt{y}}\,(\times L),(\times R)}{\Gamma,\mathtt{x}\approx\mathtt{x'} \vdash \mathtt{x'y},\mathtt{x'}\times\mathtt{y'} :: \mathtt{x'y'},\mathtt{x}\times\mathtt{y}}\,(\approx L)}{\Gamma,\mathtt{x}\approx\mathtt{x'},\mathtt{y}\approx\mathtt{y'} \vdash \mathtt{x'}\times\mathtt{y'} :: \mathtt{x}\times\mathtt{y}}\,(\approx L)}{\Gamma,\mathtt{x}\approx\mathtt{x'},\mathtt{y}\approx\mathtt{y'} \vdash \mathtt{x'}\times\mathtt{y'} \approx \mathtt{x}\times\mathtt{y},\Delta}\,(\approx R)$$

Term sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\ ::}\,(Ax_{::})}{\vdash \mathtt{x0} ::}\,(\mathtt{0}L)}{\vdash \mathtt{x0} :: \mathtt{0}}\,(\mathtt{0}R)}{\vdash \mathtt{x}\times\mathtt{0} :: \mathtt{0}}\,(\times L)}{\vdash \mathtt{x}\times\mathtt{0} \approx \mathtt{0}}\,(\approx R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z} :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(Ax_{::})}{\vdash \mathtt{x},(\mathtt{y}+\mathtt{z}) :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(+L)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(+L)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: (\mathtt{x}+\mathtt{y}),\mathtt{z}}\,(+R)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: (\mathtt{x}+\mathtt{y})+\mathtt{z}}\,(+R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \mathtt{xy},\mathtt{x'y},\mathtt{x'y'} :: \mathtt{x'y},\mathtt{x'y'},\mathtt{xy}}}{\Gamma \vdash \mathtt{xy},\mathtt{x'y},\mathtt{x'}\times\mathtt{y'} :: \mathtt{x'y},\mathtt{x'y'},\mathtt{x}\times\mathtt{y}}\,(\times L),(\times R)}{\Gamma,\mathtt{x}\approx\mathtt{x'} \vdash \mathtt{x'y},\mathtt{x'}\times\mathtt{y'} :: \mathtt{x'y'},\mathtt{x}\times\mathtt{y}}\,(\approx L)}{\Gamma,\mathtt{x}\approx\mathtt{x'},\mathtt{y}\approx\mathtt{y'} \vdash \mathtt{x'}\times\mathtt{y'} :: \mathtt{x}\times\mathtt{y}}\,(\approx L)}{\Gamma,\mathtt{x}\approx\mathtt{x'},\mathtt{y}\approx\mathtt{y'} \vdash \mathtt{x'}\times\mathtt{y'} \approx \mathtt{x}\times\mathtt{y},\Delta}\,(\approx R)$$

Term sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\ ::}\,(Ax_{::})}{\vdash \mathtt{x0} ::}\,(\mathtt{0}L)}{\vdash \mathtt{x0} :: \mathtt{0}}\,(\mathtt{0}R)}{\vdash \mathtt{x}\times\mathtt{0} :: \mathtt{0}}\,(\times L)}{\vdash \mathtt{x}\times\mathtt{0} \approx \mathtt{0}}\,(\approx R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z} :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(Ax_{::})}{\vdash \mathtt{x},(\mathtt{y}+\mathtt{z}) :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(+L)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(+L)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: (\mathtt{x}+\mathtt{y}),\mathtt{z}}\,(+R)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: (\mathtt{x}+\mathtt{y})+\mathtt{z}}\,(+R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \mathtt{xy},\mathtt{x}'\mathtt{y},\mathtt{x}'\mathtt{y}' :: \mathtt{x}'\mathtt{y},\mathtt{x}'\mathtt{y}',\mathtt{xy}}}{\Gamma \vdash \mathtt{xy},\mathtt{x}'\mathtt{y},\mathtt{x}'\times\mathtt{y}' :: \mathtt{x}'\mathtt{y},\mathtt{x}'\mathtt{y}',\mathtt{x}\times\mathtt{y}}\,(\times L),(\times R)}{\Gamma,\mathtt{x}\approx\mathtt{x}' \vdash \mathtt{x}'\mathtt{y},\mathtt{x}'\times\mathtt{y}' :: \mathtt{x}'\mathtt{y}',\mathtt{x}\times\mathtt{y}}\,(\approx L)}{\Gamma,\mathtt{x}\approx\mathtt{x}',\mathtt{y}\approx\mathtt{y}' \vdash \mathtt{x}'\times\mathtt{y}' :: \mathtt{x}\times\mathtt{y}}\,(\approx L)}{\Gamma,\mathtt{x}\approx\mathtt{x}',\mathtt{y}\approx\mathtt{y}' \vdash \mathtt{x}'\times\mathtt{y}' \approx \mathtt{x}\times\mathtt{y},\Delta}\,(\approx R)$$

Term sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash\ ::}\,(Ax_{::})}{\vdash \mathtt{x0} ::}\,(\mathtt{0}L)}{\vdash \mathtt{x0} :: \mathtt{0}}\,(\mathtt{0}R)}{\vdash \mathtt{x}\times\mathtt{0} :: \mathtt{0}}\,(\times L)}{\vdash \mathtt{x}\times\mathtt{0} \approx \mathtt{0}}\,(\approx R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z} :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(Ax_{::})}{\vdash \mathtt{x},(\mathtt{y}+\mathtt{z}) :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(+L)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: \mathtt{x},\mathtt{y},\mathtt{z}}\,(+L)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: (\mathtt{x}+\mathtt{y}),\mathtt{z}}\,(+R)}{\vdash \mathtt{x}+(\mathtt{y}+\mathtt{z}) :: (\mathtt{x}+\mathtt{y})+\mathtt{z}}\,(+R)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \mathtt{xy},\mathtt{x}'\mathtt{y},\mathtt{x}'\mathtt{y}' :: \mathtt{x}'\mathtt{y},\mathtt{x}'\mathtt{y}',\mathtt{xy}}}{\Gamma \vdash \mathtt{xy},\mathtt{x}'\mathtt{y},\mathtt{x}'\times\mathtt{y}' :: \mathtt{x}'\mathtt{y},\mathtt{x}'\mathtt{y}',\mathtt{x}\times\mathtt{y}}\,(\times L),(\times R)}{\Gamma,\mathtt{x}\approx\mathtt{x}' \vdash \mathtt{x}'\mathtt{y},\mathtt{x}'\times\mathtt{y}' :: \mathtt{x}'\mathtt{y}',\mathtt{x}\times\mathtt{y}}\,(\approx L)}{\Gamma,\mathtt{x}\approx\mathtt{x}',\mathtt{y}\approx\mathtt{y}' \vdash \mathtt{x}'\times\mathtt{y}' :: \mathtt{x}\times\mathtt{y}}\,(\approx L)}{\Gamma,\mathtt{x}\approx\mathtt{x}',\mathtt{y}\approx\mathtt{y}' \vdash \mathtt{x}'\times\mathtt{y}' \approx \mathtt{x}\times\mathtt{y},\Delta}\,(\approx R)$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),1 :: \mathtt{x},\mathtt{y},z,z',1,1}(1L),(1R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),1 :: \mathtt{x}{+}(z{+}1),\mathtt{y},z',1}(+R)}{\mathtt{x}{+}(z{+}1)\approx\mathtt{y}\vdash \mathtt{x},z,z',(),1 :: \mathtt{y},z',1}(\approx L)}{\mathtt{x}{+}(z{+}1)\approx\mathtt{y}\vdash \mathtt{x},z,z',(),1 :: \mathtt{y}{+}(z'{+}1)}(+R),(+R)}{\mathtt{x}{+}(z{+}1)\approx\mathtt{y},\ \mathtt{y}{+}(z'{+}1)\approx\mathtt{x}\vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}1)\approx\mathtt{y},\ \mathtt{y}{+}(z'{+}1\approx\mathtt{x})\vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}1)\approx\mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}1)\approx\mathtt{x})\vdash}(\exists L)}{\mathtt{x}\prec\mathtt{y},\ \mathtt{y}\prec\mathtt{x}\vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,1,1 :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,1,1}(1L),(1R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,1,1 :: \mathtt{x}{+}(\mathtt{v}_1{+}1),\mathtt{y}{+}(\mathtt{v}_2{+}1),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}1)\approx\mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1)\approx\mathtt{z}\vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,1,1 :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}1)\approx\mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1)\approx\mathtt{z}\vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}1){+}1) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}1)\approx\mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1)\approx\mathtt{z}\vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}1){+}1)\approx\mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}1)\approx\mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1)\approx\mathtt{z}\vdash \exists\mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}1)\approx\mathtt{z})}(\exists R)}{\exists\mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}1)\approx\mathtt{y}),\exists\mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}1)\approx\mathtt{z})\vdash \exists\mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}1)\approx\mathtt{z})}(\exists L)}{\mathtt{x}\prec\mathtt{y},\ \mathtt{y}\prec\mathtt{z}\vdash \mathtt{x}\prec\mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}1),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}1) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}1) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}1)}(+R),(+R)}{\mathtt{x}{+}(z{+}1) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}1) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}1) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}1 \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}1) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}1) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}1),\mathtt{y}{+}(\mathtt{v}_2{+}1),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}1) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}1) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}1){+}1) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}1) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}1){+}1) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}1) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}1) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}1) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}1) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}1) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}1),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}1) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}1) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}1)}(+R),(+R)}{\mathtt{x}{+}(z{+}1) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}1) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}1) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}1 \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}1) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}1) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}1),\mathtt{y}{+}(\mathtt{v}_2{+}1),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}1) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}1) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}1){+}1) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}1) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}1){+}1) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}1) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}1) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}1) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}1) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}1) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\bot)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,1,1 :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,1,1}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,1,1 :: \mathtt{x}{+}(\mathtt{v}_1{+}1),\mathtt{y}{+}(\mathtt{v}_2{+}1),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}1) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,1,1 :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}1) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}1){+}1) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}1) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}1){+}1) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}1) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}1) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}1) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}1) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}1) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}1) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}}$$

Sentence sequent derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},z,z',(),() :: \mathtt{x},\mathtt{y},z,z',(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x},\mathtt{y},z,z',\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},z,z',(),\mathtt{1} :: \mathtt{x}{+}(z{+}\mathtt{1}),\mathtt{y},z',\mathtt{1}}(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y},z',\mathtt{1}}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y} \vdash \mathtt{x},z,z',(),\mathtt{1} :: \mathtt{y}{+}(z'{+}\mathtt{1})}(+R),(+R)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x} \vdash z,z',() ::}(\approx L)}{\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y},\ \mathtt{y}{+}(z'{+}\mathtt{1} \approx \mathtt{x}) \vdash}(T_\perp)}{\exists z(\mathtt{x}{+}(z{+}\mathtt{1}) \approx \mathtt{y}),\ \exists z'(\mathtt{y}{+}(z'{+}\mathtt{1}) \approx \mathtt{x}) \vdash}(\exists L)}{\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{x} \vdash}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),() :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,(),()}(Ax_{::})}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1}}(\mathtt{1}L),(\mathtt{1}R)}{\vdash \mathtt{x},\mathtt{y},\mathtt{z},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}),\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}),\mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x},\mathtt{v}_1,\mathtt{v}_2,\mathtt{1},\mathtt{1} :: \mathtt{z}}(\approx L)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) :: \mathtt{z}}(+R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \mathtt{x}{+}((\mathtt{v}_1{+}\mathtt{v}_2{+}\mathtt{1}){+}\mathtt{1}) \approx \mathtt{z}}(\approx R)}{\mathtt{x}{+}(\mathtt{v}_1{+}\mathtt{1}) \approx \mathtt{y},\mathtt{y}{+}(\mathtt{v}_2{+}\mathtt{1}) \approx \mathtt{z} \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists R)}{\exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{y}),\exists \mathtt{v}(\mathtt{y}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z}) \vdash \exists \mathtt{v}(\mathtt{x}{+}(\mathtt{v}{+}\mathtt{1}) \approx \mathtt{z})}(\exists L)$$

$$\mathtt{x} \prec \mathtt{y},\ \mathtt{y} \prec \mathtt{z} \vdash \mathtt{x} \prec \mathtt{z}$$

Properties of term sequent proof theory

Definition: If $\Theta = \overline{t}_1, \ldots, \overline{t}_n$, write $\overline{r}\Theta$ for $\overline{r}\overline{t}_1, \ldots, \overline{r}\overline{t}_n$.

- $\Gamma \vdash \Theta_1 :: \Theta_2$ implies $\Gamma \vdash \overline{r}\Theta_1 :: \overline{r}\Theta_2$
- $\vdash \overline{r}\Theta_1 :: \overline{r}\Theta_2$ implies $\vdash \Theta_1 :: \Theta_2$ if 0 is not in $\overline{r}$
- $\nvdash \Theta, () ::$
- $\Gamma \vdash \Theta_1 :: \Theta_2$ iff $\Gamma \vdash \Theta, \Theta_1 :: \Theta, \Theta_2$
- $\Gamma \vdash A[x/t], \Delta$ and $\Gamma \vdash t :: t'$ implies $\Gamma \vdash A[x/t'], \Delta$
- $\vdash A$ or $\vdash \neg A$ for every atomic sentence A (including inequalities)
- Cut elimination, *full* cut elimination, for *any* derivation.
- $\nvdash \neg(1 \approx 0)$ and $\quad \nvdash$

Expressive power and simple extensions

- Term sequent logic for arithmetic corresponds to the following 'standard' equality axioms:

$$\forall \mathtt{xy}(\mathtt{x}{+}\mathtt{y} \approx \mathtt{y}{+}\mathtt{x}) \quad \forall \mathtt{xy}(\mathtt{x}{\times}\mathtt{y} \approx \mathtt{y}{\times}\mathtt{x}) \quad \forall \mathtt{xyz}(\mathtt{x}{+}(\mathtt{y}{+}\mathtt{z}) \approx (\mathtt{x}{+}\mathtt{y}){+}\mathtt{z})$$

$$\forall \mathtt{xyz}(\mathtt{x}{\times}(\mathtt{y}{\times}\mathtt{z}) \approx (\mathtt{x}{\times}\mathtt{y}){\times}\mathtt{z}) \quad \forall \mathtt{x}(\mathtt{1}{\times}\mathtt{x} \approx \mathtt{x}) \quad \forall \mathtt{x}(\mathtt{0}{\times}\mathtt{x} = \mathtt{0} \wedge \mathtt{0}{+}\mathtt{x} \approx \mathtt{x})$$

$$\forall \mathtt{xyz}((\mathtt{x}{\times}\mathtt{y}){+}(\mathtt{x}{\times}\mathtt{z}) \approx \mathtt{x}{\times}(\mathtt{y}{+}\mathtt{z})) \quad \forall \mathtt{x}\neg(\mathtt{x}{+}\mathtt{1} \approx \mathtt{0})$$

$$\forall \mathtt{x}_1\mathtt{x}_2\mathtt{y}_1\mathtt{y}_2\mathtt{z}(\mathtt{x}_1 \approx \mathtt{x}_2 \supset ((\mathtt{z}{\times}\mathtt{x}_1){+}\mathtt{y}_1 \approx (\mathtt{z}{\times}\mathtt{x}_2){+}\mathtt{y}_2 \supset \mathtt{y}_1 \approx \mathtt{y}_2))$$

- Cut-elimination can be extended with relatively little effort.

$$\frac{\Gamma, \mathtt{t}_2 \approx \mathtt{t}_1{+}(\mathtt{y}{+}\mathtt{1}) \vdash \Delta \quad \Gamma, \mathtt{t}_1 \approx \mathtt{t}_2{+}(\mathtt{y}{+}\mathtt{1}) \vdash \Delta \quad \Gamma, \mathtt{t}_1 \approx \mathtt{t}_2 \vdash \Delta}{\Gamma \vdash \Delta}\ (R_{\prec})\ \begin{array}{l}\mathtt{y} \text{ not free in}\\ \Gamma, \Delta, \mathtt{t}_1 \text{ or } \mathtt{t}_2.\end{array}$$

  allowing us to derive

$$\forall \mathtt{x}\forall \mathtt{y}((\mathtt{x} \approx \mathtt{y}) \vee (\mathtt{x} \prec \mathtt{y}) \vee (\mathtt{y} \prec \mathtt{x}))$$

  resulting in Kaye's PA^-: 'the theory of the non-negative parts of discretely ordered rings'.

Further arithmetic

- To get full arithmetic we use an infinitary rule:

$$\frac{\Gamma[\mathtt{x}/\mathtt{0}] \vdash \Delta[\mathtt{x}/\mathtt{0}] \quad \Gamma[\mathtt{x}/\mathtt{1}] \vdash \Delta[\mathtt{x}/\mathtt{1}] \quad \ldots}{\Gamma[\mathtt{x}/\mathtt{t}] \vdash \Delta[\mathtt{x}/\mathtt{t}]}\ (R_\omega)$$

  where $\mathtt{t}$ contains at most one variable.

  This yields the complete theory of arithmetic and retains cut elimination.

- Rational numbers and beyond...

$$\frac{\Gamma \vdash t\Theta_1, 1\Theta_2 :: t\Theta'_1, 1\Theta'_2}{\Gamma \vdash \Theta_1, {}^{1}\!/_{t}\Theta_2 :: \Theta'_1, {}^{1}\!/_{t}\Theta'_2}\ (\div C)$$

  $\Gamma, \Theta_2, \Theta'_2$ are not all empty

Basic syntax of the λ-calculus

- Define ***terms*** $\mathtt{t}$ and ***sentences*** $\mathtt{A}$ by:

$$\begin{array}{lll} \mathtt{t} & ::= & \mathtt{x}, \mathtt{y}, \mathtt{z}, \ldots \mid (\mathtt{t}_1 \cdot \mathtt{t}_2) \mid (\lambda \mathtt{x}.\mathtt{t}) \\ \mathtt{A} & ::= & \mathtt{t}_1 \rightsquigarrow \mathtt{t}_2 \mid \mathtt{A}_1 \wedge \mathtt{A}_2 \mid \neg \mathtt{A} \mid \forall \mathtt{x} \mathtt{A} \end{array}$$

- Define a ***tree*** Θ inductively by:
  - An individual term is a tree.
  - If Θ_1 and Θ_2 are trees then so is the ordered pair $\langle \Theta_1, \Theta_2 \rangle$.
- A ***term sequent*** is a tuple $\Gamma \vdash \Theta :\!- \mathtt{t}$.
  A ***sentence sequent*** is a pair $\Gamma \vdash \Delta$.

Term sequent rules

$$\frac{}{\Gamma \vdash \mathtt{x} :\!\!- \mathtt{x}}\ (Ax_{:\!\!-}) \qquad \frac{\Gamma \vdash \Theta_1 :\!\!- \mathtt{t}_1 \quad \Gamma \vdash \langle \mathtt{t}_2, \Theta_2 \rangle :\!\!- \mathtt{t}}{\Gamma, \mathtt{t}_1 \rightsquigarrow \mathtt{t}_2 \vdash \langle \Theta_1, \Theta_2 \rangle :\!\!- \mathtt{t}}\ (\rightsquigarrow L) \qquad \frac{\Gamma \vdash \mathtt{t}_1 :\!\!- \mathtt{t}_2}{\Gamma \vdash \mathtt{t}_1 \rightsquigarrow \mathtt{t}_2, \Delta}\ (\rightsquigarrow R)$$

$$\frac{\Gamma \vdash \Theta_1 :\!\!- \mathtt{t}_1 \quad \Gamma \vdash \langle \ldots \mathtt{t}_2[\mathtt{x}/\mathtt{t}_1] \ldots \rangle :\!\!- \mathtt{t}}{\Gamma \vdash \langle \ldots \langle \lambda \mathtt{x}.\mathtt{t}_2, \Theta_1 \rangle \ldots \rangle :\!\!- \mathtt{t}}\ (\lambda L) \qquad \frac{\Gamma \vdash \langle \Theta, \mathtt{x} \rangle :\!\!- \mathtt{t}}{\Gamma \vdash \Theta :\!\!- \lambda \mathtt{x}.\mathtt{t}}\ (\lambda R)\ \begin{array}{l}\mathtt{x} \text{ not free} \\ \text{in } \Theta\end{array}$$

$$\frac{\Gamma \vdash \langle \ldots, \langle \ldots, \mathtt{t}_1, \mathtt{t}_2 \rangle, \ldots \rangle :\!\!- \mathtt{t}}{\Gamma \vdash \langle \ldots, \langle \ldots, \mathtt{t}_1 \cdot \mathtt{t}_2 \rangle, \ldots \rangle :\!\!- \mathtt{t}}\ (\cdot L) \qquad \frac{\Gamma \vdash \Theta_1 :\!\!- \mathtt{t}_1 \quad \Gamma \vdash \Theta_2 :\!\!- \mathtt{t}_2}{\Gamma \vdash \langle \Theta_1, \Theta_2 \rangle :\!\!- \mathtt{t}_1 \cdot \mathtt{t}_2}\ (\cdot R)$$

- Sentence sequent rules are as before (replacing :: with :–).

Some derivations

- ***Fact:*** For any $\mathtt{t}$ we can derive $\Gamma \vdash \mathtt{t} \coloneq \mathtt{t}$.

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{y} \coloneq \mathtt{y}}\,(Ax_{\coloneq}) \quad \dfrac{}{\vdash \mathtt{t[x/y]} \coloneq \mathtt{t[x/y]}}}{\vdash \lambda\mathtt{x.t}, \mathtt{y} \coloneq \mathtt{t[x/y]}}\,(\lambda L)}{\dfrac{\vdash \lambda\mathtt{x.t} \coloneq \lambda\mathtt{y.(t[x/y])}}{\vdash \lambda\mathtt{x.t} \rightsquigarrow \lambda\mathtt{y.(t[x/y])}}\,(\rightsquigarrow R)}\,(\lambda R)$$

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x} \coloneq \mathtt{x}}\,(Ax_{\coloneq}) \quad \dfrac{}{\vdash \mathtt{t[x/y][y/x]} \coloneq \mathtt{t}}}{\vdash \lambda\mathtt{y.(t[x/y])}, \mathtt{x} \coloneq \mathtt{t}}\,(\lambda L)}{\dfrac{\vdash \lambda\mathtt{y.(t[x/y])} \coloneq \lambda\mathtt{y.(t[x/y])}}{\vdash \lambda\mathtt{y.(t[x/y])} \rightsquigarrow \lambda\mathtt{y.(t[x/y])}}\,(\rightsquigarrow R)}\,(\lambda R)$$

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x} \coloneq \mathtt{x}}\,(Ax_{\coloneq}) \quad \dfrac{}{\vdash \mathtt{t[x/t']} \coloneq \mathtt{t[x/t']}}}{\vdash (\lambda\mathtt{x.t}), \mathtt{t'} \coloneq \mathtt{t[x/t']}}\,(\lambda L)}{\dfrac{\vdash (\lambda\mathtt{x.t})\cdot\mathtt{t'} \coloneq \mathtt{t[x/t']}}{\vdash (\lambda\mathtt{x.t})\cdot\mathtt{t'} \rightsquigarrow \mathtt{t[x/t']}}\,(\rightsquigarrow R)}\,(\cdot L)$$

$$\dfrac{\dfrac{\dfrac{}{\vdash \mathtt{x} \coloneq \mathtt{x}}\,(Ax_{\coloneq}) \quad \dfrac{}{\vdash \mathtt{t} \coloneq \mathtt{t}}}{\vdash \mathtt{t}, \mathtt{x} \coloneq \mathtt{t}\cdot\mathtt{x}}\,(\cdot RL)}{\dfrac{\vdash \mathtt{t} \coloneq \lambda\mathtt{x.}(\mathtt{t}\cdot\mathtt{x})}{\vdash \mathtt{t} \rightsquigarrow \lambda\mathtt{x.}(\mathtt{t}\cdot\mathtt{x})}\,(\rightsquigarrow R)}\,(\cdot R)$$

More complex derivations

$$
\dfrac{
\dfrac{}{\vdash \mathtt{x} \mathrel{:\!-} \mathtt{x}}\,(Ax_{:-})
\qquad
\dfrac{
\dfrac{
\dfrac{}{\vdash \mathtt{y} \mathrel{:\!-} \mathtt{y}}\,(Ax_{:-})
\qquad
\dfrac{
\dfrac{}{\vdash \mathtt{x} \mathrel{:\!-} \mathtt{x}}\,(Ax_{:-})
\quad
\dfrac{}{\vdash \mathtt{x} \mathrel{:\!-} \mathtt{x}}\,(Ax_{:-})
}{\vdash \mathtt{x}, \mathtt{x} \mathrel{:\!-} \mathtt{x}\cdot\mathtt{x}}\,(\cdot R)
}{\vdash \mathtt{y}, \mathtt{x}\cdot\mathtt{x} \mathrel{:\!-} \mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})}\,(\cdot L)
}{\vdash \mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x}) \mathrel{:\!-} \mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})}\,(\cdot L)
}{
\dfrac{\vdash \lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big), \mathtt{x} \mathrel{:\!-} \mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})}{\vdash \lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big) \mathrel{:\!-} \lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big)}\,(\lambda R)
}\,(\lambda L)
$$

$$\vdots \; (\cdot L), (\cdot R)$$

$$\vdash \lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big)\cdot\lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big) \mathrel{:\!-} \lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big)\cdot\lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big)$$

$$\vdots \; (\cdot L), (\cdot R)$$

$$
\dfrac{\vdash \mathtt{y}\cdot\big[\lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big)\cdot\lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big)\big] \mathrel{:\!-} \mathtt{y}\cdot\big[\lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big)\cdot\lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big)\big]}{\vdash \lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big), \lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big) \mathrel{:\!-} \mathtt{y}\cdot\big[\lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big)\cdot\lambda\mathtt{x}\big(\mathtt{y}\cdot(\mathtt{x}\cdot\mathtt{x})\big)\big]}\,(\lambda L)
$$

Boolean algebra

- Define ***terms*** $\mathtt{t}$ and ***sentences*** $\mathtt{A}$ by:

$$\begin{array}{lll} \mathtt{t} & ::= & \mathtt{x}, \mathtt{y}, \mathtt{z}, \ldots \mid (\mathtt{t}_1 \cap \mathtt{t}_2) \mid \mathtt{t}^c \mid \varnothing \\ \mathtt{A} & ::= & \mathtt{t}_1 \subseteq \mathtt{t}_2 \mid \mathtt{A}_1 \wedge \mathtt{A}_2 \mid \neg \mathtt{A} \mid \forall \mathtt{x} \mathtt{A} \end{array}$$

- Let Θ range over *sets* of terms.
- A ***term sequent*** is a tuple $\Gamma \vdash \Theta :\!\!- \Theta'$.
  A ***sentence sequent*** is a pair $\Gamma \vdash \Delta$.

Term sequent rules

$$\frac{}{\Gamma \vdash \Theta_1, \Theta \coloneq \Theta, \Theta_2}\ (Ax) \quad \Theta, \Theta_1, \Theta_2 \text{ atomic} \qquad \frac{\Gamma \vdash \mathtt{t}_1, \mathtt{t}_2, \Theta_1 \coloneq \Theta_2}{\Gamma, \mathtt{t}_1 \subseteq \mathtt{t}_2 \vdash \mathtt{t}_1, \Theta_1 \coloneq \Theta_2}\ (\subseteq L) \qquad \frac{\Gamma \vdash \mathtt{t}_1 \coloneq \mathtt{t}_2}{\Gamma \vdash \mathtt{t}_1 \subseteq \mathtt{t}_2, \Delta}\ (\subseteq R)$$

$$\frac{\Gamma \vdash \Theta \coloneq \mathtt{t}, \Theta'}{\Gamma \vdash \Theta, \mathtt{t}^c \coloneq \Theta'}\ ({}^cL) \qquad \frac{\Gamma \vdash \Theta', \mathtt{t} \coloneq \Theta}{\Gamma \vdash \Theta' \coloneq \mathtt{t}^c, \Theta}\ ({}^cR) \qquad \frac{\Gamma \vdash \Theta, \mathtt{t} \coloneq \Theta'}{\Gamma \vdash \Theta, \varnothing \coloneq \Theta'}\ (\varnothing L)$$

$$\frac{\Gamma \vdash \Theta, \mathtt{t}_i \coloneq \Theta'}{\Gamma \vdash \Theta, \mathtt{t}_1 \cap \mathtt{t}_2 \coloneq \Theta'}\ (\cap L) \quad 1 \leqslant i \leqslant 2 \qquad \frac{\Gamma \vdash \Theta' \coloneq \Theta, \mathtt{t}_1 \quad \Gamma \vdash \Theta' \coloneq \Theta, \mathtt{t}_2}{\Gamma \vdash \Theta' \coloneq \Theta, \mathtt{t}_1 \cap \mathtt{t}_2}\ (\cap R)$$

Conclusions

- Term sequents allow us to manipulate terms in the same way we can manipulate propositions.
- In doing this term sequents allow us to embed a logic of terms within a logic of propositions. This allows us to reason conditionally on relationships between terms.
- Term sequent formulations of theories such as arithmetic give us the useful tool of *full* cut elimination.
- From the philosophical side, because of cut elimination, term sequents show that Arithmetic, λ-calculus, Boolean algebra etc. *really are logics* and not theories within logic.