Stone duality for first-order logic, a nominal approach

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Stone duality

I'm giving this talk to a group of people who know a lot more about duality thank I do. I welcome questions and suggestions.

Duality connects logic with topology. Take a lattice-flavoured structure; dualise it to a topology-flavoured structure.

I will present this applied to a nominal axiomatisation of first-order logic. Interesting for two reasons:

- The result is nice, and the results concrete (sets-based).
- A family of similar results is clearly possible.

The key idea is: the topology maps to its set of clopens (or equivalent), and the lattice maps to its set of maximal filters (points p) with topology generated by $\{p \mid x \in p\}$.

Nominal sets, nominal algebra

Used by Fraenkel and Mostowski to prove the independence of AC from the other axioms of set theory (circa 1930s). Applied by Gabbay and Pitts to model inductive syntax-with-binding (1999-2001).

Developed by Fernández and Gabbay (2004), then Gabbay and Mathijssen (2005-2006), into algebraic logic in which substitution, quantifiers, and λ-calculus and first-order logic were axiomatised.

Recent work by Gabbay (2011-) on nominal sets-based models of these axioms includes this presentation and two papers—one published in the Barringer Festschrift, the other submitted for publication:

- Stone duality for first-order logic: a nominal approach to logic and topology.
- Semantics out of context: nominal absolute denotations for logic and computation.

Nominal sets

This talk presents elements from both papers; primarily the first.

Fix a countably infinite collection of atoms or urelemente $\mathbb{A}$.
$a, b, c, \ldots$ range over distinct atoms.

A nominal set is a pair $\mathcal{X} = (|\mathcal{X}|, \cdot)$ of an underlying set $|\mathcal{X}|$ and a finitely-supported permutation action (for every $x \in |\mathcal{X}|$ there exists finite $A \subseteq \mathbb{A}$ such that for every permutation π if $\forall a{\in}A.\pi(a){=}a$ implies $\pi{\cdot}x = x$).

Theorem: If x has finite support A then it has a least finite supporting set $\mathrm{supp(x)} \subseteq \mathbb{A}$.

Write $a\#x$ when $a \notin \mathrm{supp(x)}$.

A function $f : \mathcal{X} \to \mathcal{Y}$ is equivariant when $f(\pi{\cdot}x) = \pi{\cdot}f(x)$.

Nominal sets (again)

The category of nominal sets and equivariant functions between them is a Boolean topos and is equal to the Schanuel topos.

A nominal set is a set $\mathcal{X}$ whose elements x 'contain' atoms.

x behaves as if it has 'free atoms' supp(x). If π fixes the 'free atoms' of x then $\pi{\cdot}x = x$.

It is important to note that supp(x) is not the same thing as the atoms contained in the structure of x. For instance, $a \notin \mathrm{supp}(\mathbb{A}) = \varnothing$ yet $a \in \mathbb{A}$.

Support measures (a)symmetry, not membership.

Nominal sets are a universe for doing finitely asymmetric mathematics over a set of indiscernibles called atoms.

Now for some miracles:

Termlike σ-algebra

A termlike σ-algebra is a tuple $\mathcal{U} = (|\mathcal{U}|, \cdot, \mathsf{sub}, \mathsf{atm})$ where:

- $(|\mathcal{U}|, \cdot)$ is a nominal set,
- an equivariant σ-action $\mathsf{sub} : \mathcal{U} \times \mathbb{A} \times \mathcal{U} \to \mathcal{U}$, written infix $v[a{\mapsto}u]$; and
- an equivariant injection $\mathsf{atm} : \mathbb{A} \to \mathcal{U}$, usually written invisibly (so we write atm(a) just as a),

such that:

$$\begin{array}{llr}
(\sigma\mathbf{id}) & & x[a{\mapsto}a] = x \\
(\sigma\#) & a\#x \Rightarrow & x[a{\mapsto}u] = x \\
(\sigma\alpha) & b\#x \Rightarrow & x[a{\mapsto}u] = ((b\ a)\cdot x)[b{\mapsto}u] \\
(\sigma\sigma) & a\#v \Rightarrow & x[a{\mapsto}u][b{\mapsto}v] = x[b{\mapsto}v][a{\mapsto}u[b{\mapsto}v]]
\end{array}$$

σ-algebra

σ-algebra captures capture-avoiding substitution—axiomatically.

It is not unusual to see substitution on FM sets. However, the usual definition would be $X[a{\mapsto}u] = \{x[a{\mapsto}u] \mid x \in X\}$.

This is not capture-avoiding. For instance, it does not satisfy ($\sigma\#$).

$a\#\mathbb{A}$ but (with the incorrect definition above)
$\mathbb{A}[a{\mapsto}b] = \mathbb{A}\setminus\{b\} \neq \mathbb{A}$.

So a σ-algebra is a non-trivial definition.

σ-algebra

A σ-algebra over a termlike σ-algebra $\mathcal{U}$ is a tuple $\mathcal{V} = (|\mathcal{V}|, \cdot, \mathsf{sub})$ where:

- $(|\mathcal{V}|, \cdot)$ is a nominal set; and
- an equivariant σ-action sub : $\mathcal{V} \times \mathbb{A} \times \mathcal{U} \to \mathcal{V}$, written infix $v[a{\mapsto}u]$

such that:

$$
\begin{array}{llrl}
(\sigma\mathbf{id}) & & x[a{\mapsto}a] &= x \\
(\sigma\#) & a\#x \Rightarrow & x[a{\mapsto}u] &= x \\
(\sigma\alpha) & b\#x \Rightarrow & x[a{\mapsto}u] &= ((b\ a)\cdot x)[b{\mapsto}u] \\
(\sigma\sigma) & a\#v \Rightarrow & x[a{\mapsto}u][b{\mapsto}v] &= x[b{\mapsto}v][a{\mapsto}u[b{\mapsto}v]]
\end{array}
$$

FOL algebra

Suppose $\mathcal{U} = (|\mathcal{U}|, \cdot, \mathsf{sub}, \mathsf{atm})$ is a termlike σ-algebra.

A FOL-algebra over $\mathcal{U}$ is a poset $\mathcal{L} = (\mathcal{X}, \leq)$ in nominal sets, with fresh-finite limits, a complement, and a compatible σ-action.

Given $X \subseteq |\mathcal{X}|$ its $A\#$limit is a greatest element $\bigwedge^{\#A} X$ such that $A \cap \mathsf{supp}(\bigwedge^{\#A}\mathsf{X}) = \varnothing$ and $\bigwedge^{\#A} X \leq x$ for every $x \in X$.

Then $\top$ is the $\varnothing\#$limit of $\varnothing$.

$x \wedge y$ is the $\varnothing\#$limit of $\{x, y\}$.

$\forall a.x$ is the $\{a\}\#$limit of $\{x\}$. $\Leftarrow$ amazing!

A σ-action is compatible when if $a \notin A$ and $A \cap \mathsf{supp}(\mathsf{u}) = \varnothing$ then

$$(\bigwedge^{\#A} X)[a \mapsto u] = \bigwedge^{\#A}\{x[a \mapsto u] \mid x \in X\} \text{ and } \neg(x[a \mapsto u]) = (\neg x)[a \mapsto u].$$

FOL algebra

FOL algebras are easily axiomatised in nominal algebra. See the paper.

Example axioms:

$$\forall a.x \leq x \quad \text{and} \quad a\#y \Rightarrow \forall a.(x \vee y) = (\forall a.x) \vee y$$

(Equation on the right looks quasiequational, but actually it's not. This is a fundamental observation about nominal algebra due to Gabbay and Mathijssen.)

In the case that the FOL algebra is built out of sets (e.g. is clopens of some suitable topology) then $\forall a.O$ ($\{a\}$-fresh limit of $\{O\}$) is equal to $\bigcap_{u \in |\mathcal{U}|} O[a \mapsto u]$. That is, given a compatible σ-action

$$\forall a.O = \bigcap_{u \in |\mathcal{U}|} O[a \mapsto u] \quad \Leftarrow \text{ amazing!}$$

(σ-action on O given in 3 slides.)

Duality: what you'd expect

It's what you expect.

Nominal poset $\mathcal{L}$ maps to maximal filters p, topologised by $\{p \mid x \in p\}$.

A topology maps to its set of clopens.

What you'd not expect

The σ-action on $\mathcal{L}$ dualises on the topological side to an amgis-action ($\reflectbox{$\sigma$}$-action).

An $\reflectbox{$\sigma$}$-algebra over $\mathcal{U}$ is a tuple $\mathcal{P} = (|\mathcal{P}|, \cdot, \reflectbox{σ}, \mathcal{U})$ of an underlying nominal set $(|\mathcal{P}|, \cdot)$, and an amgis-action $\reflectbox{$\sigma$} : |\mathcal{P}| \times \mathbb{A} \times |\mathcal{U}| \to |\mathcal{P}|$ written $p[u \mapsfrom a]$, such that:

$$
\begin{array}{lrcl}
(\reflectbox{σ}\mathbf{id}) & p[a \mapsfrom a] & = & p \\
(\reflectbox{σ}\sigma) & a\#v \Rightarrow p[v \mapsfrom b][u \mapsfrom a] & = & p[u[b \mapsto v] \mapsfrom a][v \mapsfrom b]
\end{array}
$$

Contrast with σ-axioms:

$$
\begin{array}{llrcl}
(\sigma\mathbf{id}) & & x[a \mapsto a] & = & x \\
(\sigma\#) & a\#x \Rightarrow & x[a \mapsto u] & = & x \\
(\sigma\alpha) & b\#x \Rightarrow & x[a \mapsto u] & = & ((b\ a)\cdot x)[b \mapsto u] \\
(\sigma\sigma) & a\#v \Rightarrow & x[a \mapsto u][b \mapsto v] & = & x[b \mapsto v][a \mapsto u[b \mapsto v]]
\end{array}
$$

What you'd not expect

The topology acquires extra properties.

Open sets have a σ-action generated as functional preimage using the ʊ-action:

$$O[a\mapsto u] = \{p \mid p[u\hookleftarrow a] \in O\}.$$

The topology must have the following closure properties:

- If O is open then so is $\forall a.O$.
- If $p \in O$ and $a\#p$ then $p \in \forall a.O$ (dualised $\forall$-right rule!).
- Every $\exists a$-cover has a finite subcover, where a $\exists a$-cover is a cover $\mathcal{U}$ such that if $O \in \mathcal{U}$ and $a\#\mathcal{U}$ then also $\exists a.O \in \mathcal{U}$.

Go figure.

What is this maths telling us?

Replay for other logics. Replay for λ-calculus; independence-friendly logic; whatever quantifier you can imagine.

(One previous result for the $\reflectbox{N}$-quantifier with Petrişan, 2011.)

Radically different approach to names than hyperdoctrines and the like. But not incompatible: nominal sets admit a presheaf presentation, indexed by finite sets of atoms. A nominal element x exists at all A such that $\mathrm{supp(x)} \subseteq \mathrm{A}$.

What is this maths telling us?

I love the sets-based presentation. It gives a nice satisfying concrete representation.

I am fascinated by the notion of fresh-finite limit $\bigwedge^{\#A} X$. Crying out for generalisation from lattices to categories (in nominal sets).

I am also fascinated by σ-algebras. Somehow, they give an alternative to the Lawvere 'adjoints' account of quantifiers. It is not quite understood how, except (intuitively) that a σ-algebra generalises a category, in that x with $\mathrm{supp}(x) = \{a\}$ is 'like' an arrow (it can be composed with y with $\mathrm{supp}(y) = \{a\}$ by taking $x[a{\mapsto}y]$).

What is this maths telling us?

Nominal sets / algebra / posets are powerful.

They really offer new results. I am only scratching the surface here.