Consistency of Quine's NF using nominal techniques

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8 September 2016

Introduction

Thanks to the organisers. Thank you all for coming.

Bibliography:

- [gabbay:nomua]
  "Nominal universal algebra"
  (coauthored with Aad Mathijssen).
- [gabbay:semooc]
  "Semantics out of context"
- [gabbay:repdul]
  "Representation and duality of the untyped lambda-calculus"
  (coauthored with Michael J. Gabbay).
- [gabbay:conqnf]
  "Consistency of Quine's NF"

All available from `www.gabbay.org.uk/papers.html`
Type Control+F and search for nomuae, semooc, repdul, conqnf.

Introduction

These papers contain a new and interesting 'nominal' view of syntax and semantics, culminating with my claimed proof of consistency of Quine's New Foundations (ConNF).

NF is a set theoretic foundation for mathematics.

Proposed in 1937 its consistency is described by the Stanford Encyclopaedia of Philosophy as the oldest outstanding consistency question.

My claimed proof is here:
`http://www.gabbay.org.uk/papers.html#conqnf`
This document is under review.

Here's a taste of the proof:

The language of sets

Syntactic classes are

- atoms (variable symbols) $a, b, c \in \mathbb{A}$,
- terms s, and
- predicates ϕ:

$$
\begin{array}{lll}
s, t, u & ::= & a \in \mathbb{A} \mid \{a \mid \phi\} \\
\phi & ::= & \phi \land \phi \mid \neg\phi \mid \forall a.\phi \mid t \in s \mid s = t
\end{array}
$$

We call $\{a \mid \phi\}$ a comprehension.

Naivete is simple, but dangerous

Write axiom (**Ext**) for extensionality and axiom-scheme (**Comp**) for comprehension

$$\begin{array}{ll} (\textbf{Ext}) & \forall a.\forall a.(a = b \Leftrightarrow \forall c.(c \in a \Leftrightarrow c \in b)) \\ (\textbf{Comp}) & \exists a.\forall b.(b \in a \Leftrightarrow \phi) \quad \text{for any predicate } \phi \end{array}$$

then we can write

$$Naive = (\textbf{Ext}) + (\textbf{Comp}).$$

In words:

$$Naive = \text{extensionality} + \text{comprehension}.$$

This is a great theory but it proves $Russell \in Russell$ and also $Russell \notin Russell$ where $Russell = \{a \mid a \notin a\}$.

Oops!

On NF

NF refines Naive sets and has (at least) two elegant features:

1. Like naive set theory, NF admits a universal set. The set of all sets $\{a \mid \top\}$ is a set.
   Note that 'traditional' systems, such as ZF set theory, do not permit this. Nice!
2. It uses a beautiful and mysterious stratifiability condition to avoid Russell's paradox (described below).

Stratifiability

ϕ is stratifiable when there exists an assignment of an integer level to its atoms such that

- if we extend to all terms by setting $\mathit{level}(\{a \mid \phi\}) = \mathit{level}(a){+}1$, then
- if $s{=}t$ appears in ϕ then $\mathit{level}(s) = \mathit{level}(t)$, and
- if $t{\in}s$ appears in ϕ then $\mathit{level}(t){+}1 = \mathit{level}(s)$.

Let's try to stratify some comprehensions,

1. Russell's 'set',
2. the set of all sets,
3. the set of all two-element sets:

Stratifiability

$$\begin{aligned} Russell &= \{a \mid a \notin a\} \\ Univ &= \{a \mid \top\} \\ 2 &= \{d \mid \exists a, b. a{\neq}b \wedge \forall c.(c \in d \Leftrightarrow (c{=}a{\vee}c{=}b))\} \end{aligned}$$

Stratifiability

$$\mathit{Russell} = \{a^1 \mid a^0 \notin a^1\}$$

$$\mathit{Univ} = \{a^1 \mid \top\}$$

$$2 = \{d^1 \mid \exists a^0, b^0 . a^0 \neq b^0 \wedge \forall c^0 . (c^0 \in d^1 \Leftrightarrow (c^0 = a^0 \vee c^0 = b^0))\}$$

Axioms of NF

If we write axiom (**Ext**) for extensionality and axiom-scheme (**SC**) for stratifiable comprehension

$$
\begin{array}{ll}
(\mathbf{Ext}) & \forall a.\forall a.(a = b \Leftrightarrow \forall c.(c \in a \Leftrightarrow c \in b)) \\
(\mathbf{SC}) & \exists a.\forall b.(b \in a \Leftrightarrow \phi) \quad (\phi \text{ stratifiable})
\end{array}
$$

then we can write

$$NF = (\mathbf{Ext}) + (\mathbf{SC}).$$

In words:

$$NF = \text{extensionality} + \text{stratifiable comprehension}.$$

As if by magic, Russell's paradox is removed.

But what does it mean? And is NF consistent?? Read on ...

Structure of my proof

My proof falls into roughly the following parts:

1. Understand stratification.
2. Understand NF's impredicative $\forall$ — impredicative, because the universal set is a set.
3. Understand extensional equality — which uses the impredicative universal quantification.

Stratification = normal forms

Consider the following rewrite:

$$s \in \{a \mid \phi\} \to \phi[a{:=}s].$$

Sensible since s is in 'the set of a such that ϕ' iff $\phi[a{:=}s]$.

Theorem (G.): Stratifiable terms are confluent and strongly normalising: they rewrite confluently and in finite time to a unique normal form in the syntax

$$\begin{array}{lll} s, t, u & ::= & a \in \mathbb{A} \mid \{a \mid \phi\} \\ \phi & ::= & \phi \land \phi \mid \neg\phi \mid \forall a.\phi \mid t{\in}a \mid s{=}t \end{array}$$

This is the only property of stratifiability my proof needs. Thus:

Stratification = We can work with normal forms

Points = filters of normalised syntax

Our denotation $[\![\phi]\!]$ for a predicate ϕ will be a set of points.

A point p is a 'well-behaved' maximally consistent sets of predicates (ultrafilters).

It is a standard 'trick' to build logical semantics out of maximally consistent sets of predicates. What is new is to notice that stratifiability gives NF syntax normal forms, and that this implies a base case for an inductive definition:

$$[\![t\in a]\!] = \{p \in \mathit{Points} \mid (t\in a) \in p\}.$$

The use of normalisable syntax is important precisely here: without it, calculating $[\![t\in s]\!]$ might lead to an infinite chain where $p \in [\![t\in s]\!] \Leftrightarrow p \notin [\![t\in s]\!]$, as in Russell's paradox.

The well-behavedness conditions required of points are subtle: magic hidden here.

$\forall$ = nominal colimit

Following [gabbay:semooc], $\forall$ is a (nominal) fresh colimit. In lattice terminology

$$\forall a.x = \bigvee\{z \mid z \leq x,\ a\#z\}.$$

This is consistent with a similar definition for $\wedge$:

$$x \wedge y = \bigvee\{z \mid z \leq x,\ z \leq y\}.$$

More specifically for the lattice of sets of points we define:

$$[\![\forall a.\phi]\!] = \bigcup\{X \subseteq \mathit{Points} \mid X \subseteq [\![\phi]\!],\ a\#X\}$$

That is, the meaning of $\forall a.\phi$ is the greatest set of points in the meaning of ϕ for which a is nominally fresh.

Nominal $\forall$ $\neq$ Tarski $\forall$

Tarski's notion of truth would define

$$[\![\forall a.\phi]\!] = \bigcap_{t} [\![\phi[a{:=}t]]\!] \quad \text{which is not} \quad [\![\forall a.\phi]\!] = \bigcup\{X \mid X \subseteq [\![\phi]\!],\ a\#X\}.$$

The right-hand definition has the advantage of being untroubled by impredicativity:

- We can inductively calculate $[\![\forall a.\phi]\!]$ if we know $[\![\phi]\!]$, since ϕ is smaller than $\forall a.\phi$.
- In the presence of impredicative quantification, such as appears in NF, $\phi[a{:=}t]$ may be larger than ϕ. So we cannot necessarily calculate $[\![\phi[a{:=}t]]\!]$ even if we know $[\![\phi]\!]$.

Substitution as a nominal algebra

Nominal algebra [gabbay:nomuae] is like universal algebra but over nominal sets, so enriched with names, freshness, and binding.

A sigma-algebra is a set $\mathbb{S}$ with an operation σ (written infix ad -[-:=-] satisfying the following axioms:

$$\begin{aligned} a\#Z \Rightarrow \quad & Z[a\mapsto X] = Z \\ & Z[a\mapsto a] = Z \\ a\#Y \Rightarrow \quad Z[a\mapsto X][b\mapsto Y] & = Z[b\mapsto Y][a\mapsto X[b\mapsto Y]] \\ b\#Z \Rightarrow \quad & Z[a\mapsto X] = ((b\ a)\cdot Z)[b\mapsto X] \end{aligned}$$

So 'substitution' can be axiomatised in nominal sets as an operation

$$\sigma : \mathbb{S} \times \mathbb{A} \times \mathbb{T} \to \mathbb{S}$$

—just as group multiplication is axiomatised in ordinary universal algebra as $\cdot : G \times G \to G$.

Substitution is an algebra!

An interlude: substitution as a nominal algebra

Theorem (G.): normalised syntax of NF forms a sigma-algebra, where the sigma-action is given by substitution+renormalisation.

Theorem (G.): If $\mathbb{S}$ is a sigma-algebra then so is $\mathit{Powerset}(\mathit{Powerset}(\mathbb{S}))$ (proof nontrivial).

Observation:

- $[\![\phi]\!]$ is a set of points,
- points are sets of predicates,
- predicates have a sigma-action (by first theorem above), so
- the lattice of denotations of NF as sets of points, which is in $\mathit{Powerset}(\mathit{Powerset}(\mathit{predicates}))$, is also a sigma-algebra in a natural way.

Theorem (G.): The natural sigma-algebra action is compositional:

$$[\![\phi[a{:=}s]]\!] = [\![\phi]\!][a{:=}s].$$

Overview of the semantics

$$
\begin{aligned}
[\![\phi \land \psi]\!] &= [\![\phi]\!] \cap [\![\psi]\!] \\
[\![\neg\phi]\!] &= \mathit{Points} \setminus [\![\phi]\!] \\
[\![t \in a]\!] &= \{p \in \mathit{Points} \mid t \in a \in p\} \\
[\![\forall a.\phi]\!] &= \bigcup\{X' \subseteq [\![\phi]\!] \mid a\#X'\}
\end{aligned}
$$

Various propositions and lemmas, with nontrivial proofs, ensure that this all fits together to give a sound model of NF.

Equality (briefly)

One well-behavedness property of NF is that the model must be extensional. That is, we require:

$$[\![\forall a.\forall a.(a = b \Leftrightarrow \forall c.(c \in a \Leftrightarrow c \in b))]\!] = \mathit{Points}.$$

This is really a restriction on when a set of predicates p is a point: if a consistent set of predicates p does not validate extensionality then it's not well-behaved and we throw it out!

How do we know that any point exists?

Equality (briefly)

NF actually has two primitive predicate forms:

- $t \in s$.
- $s = t$.

These are equally expressive:

- $s = t$ maps to $a \in \{b \mid s = t\}$
  (for fresh dummy variables a and b) and
- $t \in s$ maps to $\{b \mid t \in s\} = \{b \mid \top\}$
  (for a fresh dummy variable a).

Extensionality 'likes' ultrafilters based on the equality predicate-form, but the rest of the paper (dealing with $\neg$, $\wedge$, $\forall$, and $\in$) 'likes' ultrafilters based on the sets membership predicate-form.

The final third of the proof

The final third of the paper shows how to move between two notions of filter, one based on $\in$ and one based on $=$.

The proof of consistency for NF then reduces to constructing a specific $=$-style filter which, when translated to $\in$-style, is a point.

This construction relies heavily on nominal techniques once again. Small support properties are used to place strong bounds on the size of small-supported powersets.

The final third of the proof

I will give just a taste of how it works: in nominal sets, both $\mathbb{A}$ the set of names and $powerset_{fs}(\mathbb{A})$ are countable; basically this is because any finitely-supported set of atoms is either finite or cofinite.

A generalisation of this result gives us that if $\mathbb{A}$ is 'large' and X is a 'large' nominal set then, in a certain sense, $powerset_{ss}(X)$ the small-supported set of subsets of X is also 'large' (and not 'larger').

In other words, small-supported powersets have the same cardinality as the original set. This use of support to control cardinalities is crucial.

This is the most technical part of the paper, and is the only part of the construction to be quite specific to consistency of NF. If interested, trace uses of Lemma 10.27 in my paper.

Conclusions

The proof of consistency of NF breaks down into distinct pieces:

1. A theory of normal forms based on stratifiability.
2. The observation that normal forms and sets of sets of normal forms form different but related sigma-algebras.
3. A theory of universal quantification $\forall$ based on fresh-finite limits in nominal lattices, differing from standard Tarski semantics, and nicely compatible with impredicativity.
4. Combinatorial arguments on nominal sets, noting that nominal powersets do not get large 'too quickly' because of support restrictions.

Each of these points is independently interesting and seems likely to be useful not just for ConNF.

Bits ’n pieces:

Go catch a very large set of atoms...

If X is a set write $\#X$ for the cardinality of X.

Let $\beth_0 = \#\mathbb{N}$. Write $\beth_\omega$ for the least cardinal larger than $\#\mathit{powerset}^n(\mathbb{N})$ for every $n \in \mathbb{N}$. So:

$$\begin{aligned} & \beth_0 = \#\mathbb{N} \\ \leq\ & \beth_1 = \#\mathit{powerset}(\mathbb{N}) \\ \leq\ & \beth_2 = \#\mathit{powerset}(\mathit{powerset}(\mathbb{N})) \leq \cdots \leq \beth_\omega. \end{aligned}$$

Fix a large (size $\beth_\omega$) set of atoms $\mathbb{A}$.

- Write $\text{И}a.\Phi(a)$ when Φ holds of all but $\kappa \lneq \beth_\omega$ many atoms. Read this as 'for new a, $\Phi(a)$'.
- Write $\text{Ə}a.\Phi(a)$ when Φ holds of $\kappa = \beth_\omega$ many atoms. Read this as 'for generously many a, $\Phi(a)$'.

Proof of confluence of syntax normalisation

Consider the rewrite on terms and predicates:

$$s \in \{a \mid \phi\} \rightarrow \phi[a \mapsto s].$$

Theorem: Stratifiable terms are confluent and strongly normalising under this rule. That is, they rewrite confluently and in finite time to a unique normal form.

Proof sketch: Confluence is routine. Termination follows by rewriting innermost highest level reducts. Use a multiset lexicographic ordering, which is well-founded.

Syntax of normal forms

We can easily characterise normal forms:

$$\begin{array}{lll} s, t, u & ::= & a \in \mathbb{A} \mid \{a \mid \phi\} \\ \phi & ::= & \phi \wedge \phi \mid \neg\phi \mid \forall a.\phi \mid t \in a \end{array}$$

Note the $t \in a$ on the far right; this is the base case of induction on normalised syntax.

Let a prepoint $p \in \text{Prepoint}$ be a set of assertions of the form $t \in a$. Then we provisionally interpret $t \in a$ by

$$[\![t \in a]\!] = \{p \in \text{Prepoint} \mid (\text{t} \in \text{a}) \in \text{p}\}.$$

Now we want to interpret $\wedge$, $\neg$, $=$, and $\forall$, in the syntax above in such a way as to validate all the axioms of NF.

This will be our model.

Overview of our model

Our model will interpret a predicate as a set of points, where a point is a prepoint plus conditions.

$$\phi \longmapsto [\![\phi]\!] \in \mathit{powerset}(\mathit{powerset}(\{t{\in}a \mid \text{all } t, a\})).$$

So how to interpret logical connectives $\wedge$, $\neg$, $=$, and $\forall$?

Logic in nominal powersets (propositional part; high-level view)

Conjunction and negation correspond to sets intersection and complement, as usual.

$$[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!] \qquad [\![\neg\phi]\!] = \mathit{Points} \setminus [\![\phi]\!]$$

(I haven't said which prepoints are points, or proved that any points exist.)

Sets membership becomes substitution, thanks to our rewrite rule:

$$[\![\{b \mid \psi\} \in \{a \mid \phi\}]\!] = [\![\phi[a \mapsto \{b \mid \psi\}]]\!].$$

(This isn't trivial to check.)

Logic in nominal powersets (quantifiers)

What about quantification $[\![\forall a.\phi]\!]$? Following [semooc,repdul] we write:

$$[\![\forall a.\phi]\!] = \{p \mid \text{И}b.(b\ a)\cdot\phi \in p\}.$$

It turns out that this has many equivalent presentations, including:

$$[\![\forall a.\phi]\!] = \bigcup\{X' \subseteq [\![\phi]\!] \mid a\#X'\}.$$

Thus $[\![\forall a.\phi]\!]$ is the greatest subset of $[\![\phi]\!]$ for which a is fresh, in the sense of nominal sets.

This characterisation of quantification uses only И and #. It does not depend on substitution!

Logic in nominal powersets (quantifiers)

This

$$[\![\forall a.\phi]\!] = \bigcup\{X' \subseteq [\![\phi]\!] \mid a\#X'\}$$

guarantees that:

$$\frac{}{[\![\forall a.\phi]\!] \subseteq [\![\phi]\!]} \qquad \frac{[\![\psi]\!] \subseteq [\![\phi]\!] \quad (a\#\psi)}{[\![\psi]\!] \subseteq [\![\forall a.\phi]\!]}$$

Note we do not use the familiar Tarski semantics that forall = 'for every possible value'. This would read as follows:

$$[\![\forall a.\phi]\!] = \bigcap_u [\![\phi[a \mapsto u]]\!].$$

That depends on substitution. We can't do that in NF because NF is impredicative and u may be a comprehension $\{a \mid \psi\}$ where ψ is larger than ϕ — taking $\phi[a \mapsto u]$ in a definition would be unhealthy for inductive quantities.

Logic in nominal powersets (quantifiers)

The nominal semantics of $\forall$ works generally, just like conjunction and complement.

If $X, Y \subseteq \mathcal{X}$ are subsets of a nominal set $\mathcal{X}$ we can define

$$\forall a.X = \bigcup \{X' \subseteq X \mid a\#X'\}$$

and then

$$\frac{}{\forall a.X \subseteq X} \qquad \frac{Y \subseteq X \quad (a\#Y)}{Y \subseteq \forall a.X}.$$

Thus $\forall a.X$ is the greatest subset of X for which a is fresh.

(This generalises further to nominal lattices; see [semooc].)

Logic in nominal powersets (quantifiers)

Given an extra consistency condition on prepoints called generous naming of internal sets we obtain a theorem (Theorem 8.15 in the paper):

$$[\![\forall a.\phi]\!] = \bigcap_u [\![\phi[a \mapsto u]]\!].$$

So by the end of my paper, $\forall$ is doing what we expect and quantifying over all terms.

It matters that this is a theorem, not a definition: the ϕ on the right is smaller than the $\forall a.\phi$ on the left in

$$[\![\forall a.\phi]\!] = \bigcup \{X' \subseteq [\![\phi]\!] \mid a\#X'\}.$$

So this is suitable for an inductive definition; the first equality above is not, in NF.

Substitution

An important lemma is that

$$[\![\phi[a\mapsto u]]\!] = [\![\phi]\!][a\mapsto u].$$

This is non-trivial to prove.

Indeed, it is also non-trivial to state. What is $[\![\phi]\!][a\mapsto u]$?

We know that $[a\mapsto u]$ applied to syntax ϕ is.

What is $[a\mapsto u]$ applied to a set of (pre)points like $[\![\phi]\!]$?

Substitution

Suppose $\mathcal{X}$ has a σ-action $x[a\mapsto u]$. Suppose $p \in \mathit{powerset}(\mathcal{X})$ and $X \in \mathit{powerset}(\mathit{powerset}(\mathcal{X}))$ $[\![\phi]\!]$ is one of these X).

Then define:

$$
\begin{aligned}
x \in p[u\mapsfrom a] &\Leftrightarrow x[a\mapsto u] \in p \\
p \in X[a\mapsto u] &\Leftrightarrow \text{И} b.(p[u\mapsfrom b] \in (b\ )\cdot X)
\end{aligned}
$$

Amgis $[u\mapsfrom a]$ is the functional preimage of underlying substitution. The σ-action on X is obtained from the amgis action on p.

Studying the two definitions above is a talk in itself. The bottom line is:

$[\![\phi]\!][a\mapsto u]$ is obtained by ‘lifting’ $\phi[a\mapsto u]$ as above.

New and Generous

A filter p generously names x when $\text{Ɔ}a.(a{=}x \in p)$, meaning that $a{=}x \in p$ for $\beth_\omega$ many atoms a.

This guarantees that if $\text{И}a.\phi(a) \in p$ then $\phi(a) \in p$ for some a such that $a{=}x \in p$

The mechanics of the proof require the set of atoms to have cardinality $\#\mathbb{A} = \beth_\omega = \bigcup_{i<\omega} \#2^i$.

We unpack this further:

- $\text{И}a.\Phi(a)$ holds when $\#\{a \mid \neg\Phi(a)\} < \beth_\omega$.
- $\text{Ɔ}a.\Phi(a)$ holds when $\#\{a \mid \Phi(a)\} = \beth_\omega$.
- И and Ɔ are dual: $\text{И}a.\Phi(a) \Leftrightarrow \neg\text{Ɔ}a.\neg\Phi(a)$.

The technical bits: equality and quantification

The technical rubber hits the mathematical road around Definition 11.19, Proposition 11.30, and Definition 12.37.

We require extensionality, that

$$\forall \phi, s, t, p. \big(p \in [\![s{=}t]\!] \Rightarrow (p \in [\![\phi[a{:=}s]]\!] \Leftrightarrow p \in [\![\phi[a{:=}t]]\!])\big).$$

We enforce this by an inductive construction to build a maximally consistent extensional set of equalities $s{=}t$.

We also require generous naming of internal sets, that for every p and s,

$$\reflectbox{E} a. \forall t. \big(p \in [\![t{\in}a]\!] \Leftrightarrow p \in [\![t{\in}s]\!]\big).$$

We enforce this by another induction generating a maximally consistent set of $(t{\in}a)$s.

About my proof

- Stratifiability gives a normal form for the rewrite $x \in \{a \mid \phi\} \to \phi[a{\mapsto}x]$.
- $\forall$ modelled using nominal limits.
- Sets extensionality handled by saturating extensionality equalities to a greatest fixedpoint.
- $\forall$-elimination $(\forall E)$ $(\forall a.\phi) \Rightarrow \phi[a{\mapsto}x]$ modelled by logical dual to И called the 'Generous' quantifier Ə. Generosity also corresponds to proof-theoretic strength.
- Semantics of predicates as sets of points, where a point is a maximally consistent set of predicates.
- Substitution modelled using σ-algebras.
- Comprehension = atoms-abstractions. So $[\![\{a \mid \phi\}]\!] = [a][\![\phi]\!]$.
- Atoms extensionally equal to, but not syntactically identical to, comprehensions: $[\![a]\!] = [\![\{b \mid b{\in}a\}]\!]$.
- Two equivalent, but structurally distinct, notions of 'maximally consistent sets': one designed for = and the other for $(\forall E)$.

Cheat-sheet

$$
\begin{aligned}
\text{Stratifiability} &= x \in \{a \mid \phi\} \to \phi[a{\mapsto}x] \text{ terminates} \\
\forall &= \text{И} \\
\exists &= \text{Ↄ} \\
\text{Extensionality} &= \text{gfp of “if } x,\, y \text{ have same elements then add } x = y\text{”} \\
(\forall\mathbf{E}) &= (\#\mathbb{A}{=}\#\text{Sets}) \\
[\![\phi]\!] &= \{p \in \mathit{points} \mid \phi \in p\} \\
[\![\phi[a{\mapsto}u]]\!] &= \{p \mid p[u{\hookleftarrow}a] \in [\![\phi]\!]\} \\
\text{Sets} &= [\mathbb{A}]\text{Predicates}
\end{aligned}
$$

$\mathbb{A}$ is the set of atoms.