Semitopology and (paraconsistent) logic

Murdoch J. Gabbay (joint work with Giuliano Losa)

Dagstuhl Seminar 23272

3 July 2023

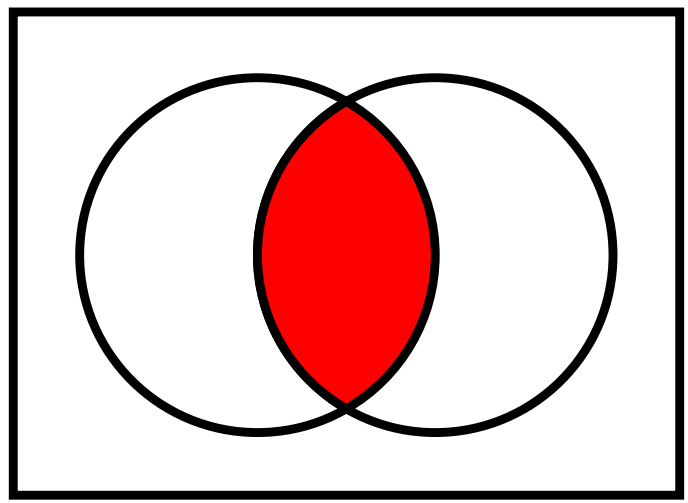
Definition of a semitopology

Definition. A semitopology is a pair $(P, \mathrm{Open} \subseteq pow(P))$ of

- P a nonempty set of points and
- Open a set of open sets such that $P \in \mathrm{Open}$ and

$$\mathcal{O} \subseteq \mathrm{Open} \Rightarrow \bigcup \mathcal{O} \in \mathrm{Open}.$$

Think: *"topology, minus condition that $\cap$ of two opens is open".*



(Image credit: Wikipedia.)

Differences from topologies

Semitopologies have their own rich mathematical structure, distinct from topologies.

Here is one simple difference:

- In topology, a *minimal* open neighbourhood of p is *least* — two minimal open neighbourhoods of p can be intersected to form a lesser open neighbourhood of p, which by minimality is equal to both.
- In a semitopology, p may have multiple minimal open neighbourhoods.

Motivation: heterogeneous consensus

- Interpret $p \in \mathsf{P}$ as a *participant*.
- Interpret an open neighbourhood $p \in O \in \mathsf{Open}$ as a *quorum*.

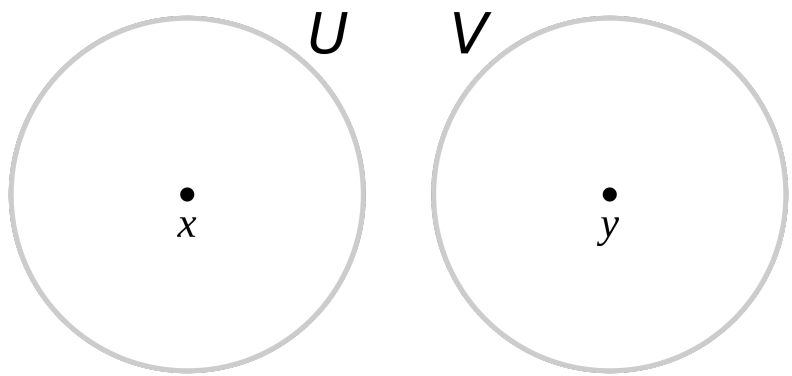
Several distributed systems, including Stellar and Ripple, can be usefully thought of as semitopologies in which participants can progress when they reach agreement on an open neighbourhood.

Hausdorff separation

Notation. Write $O \between O'$ when $O \cap O' \neq \varnothing$.

You may be familiar with the (standard) Hausdorff property that $p \neq p' \in \mathrm{P}$ have disjoint open neighbourhoods:

$$\exists O, O' \in \mathrm{Open}.(p \in O \wedge p' \in O') \wedge \neg(O \between O').$$



(Image credit: Wikipedia.)

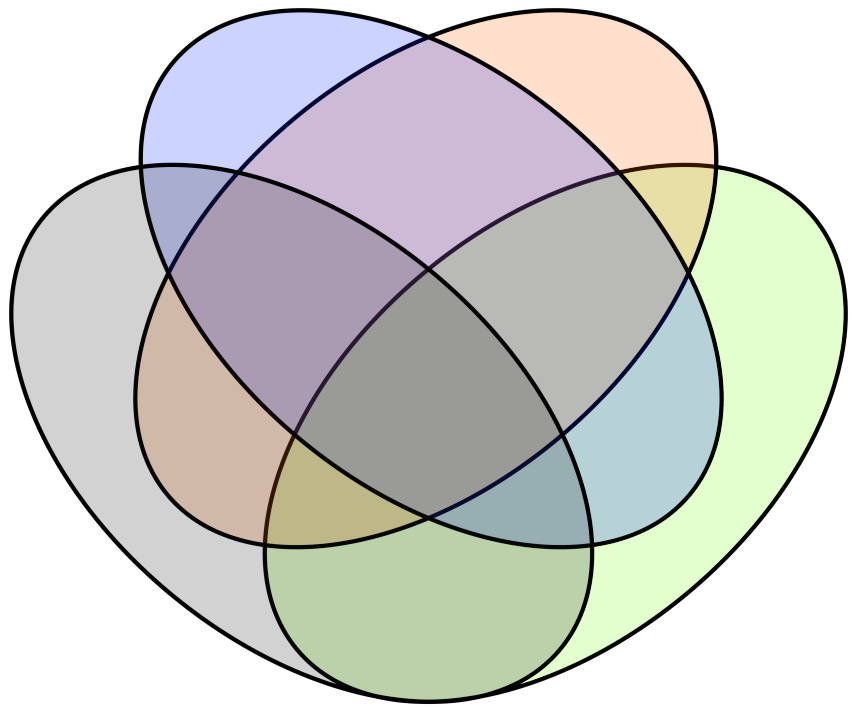
Let's consider an *anti-Hausdorff* property:

Intertwined points: the anti-Hausdorff property

Definition. Call p and p' intertwined and write $p \between p'$ when

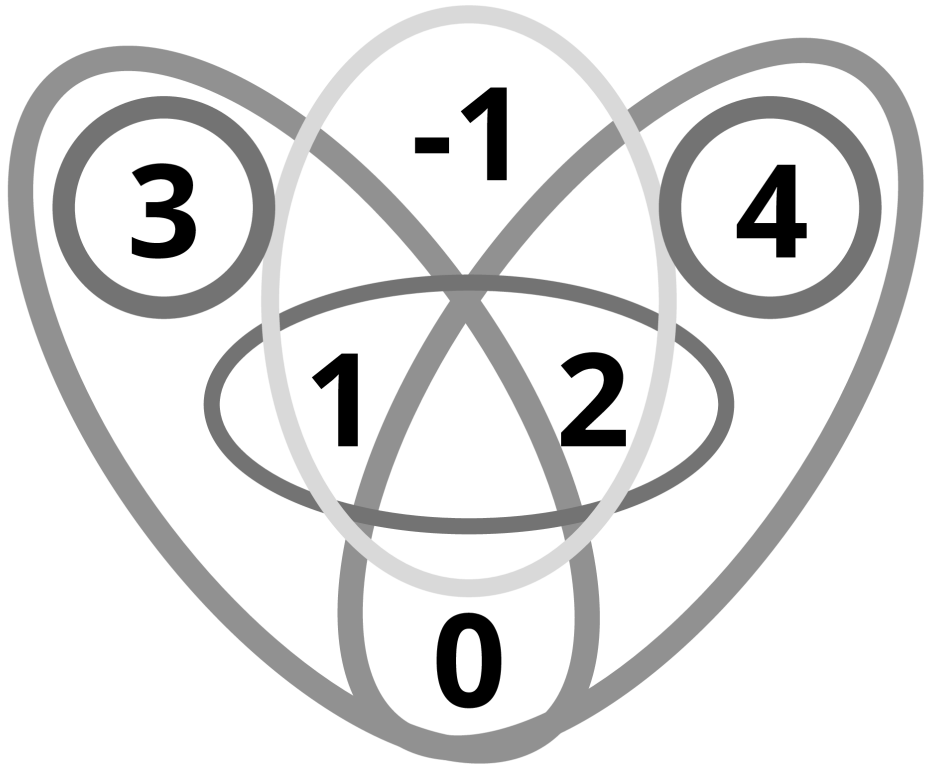
$$\forall O, O' \in \mathsf{Open}.(p \in O \wedge p' \in O') \Rightarrow O \between O'.$$

So $p \between p'$ when all their open neighbourhoods intersect.



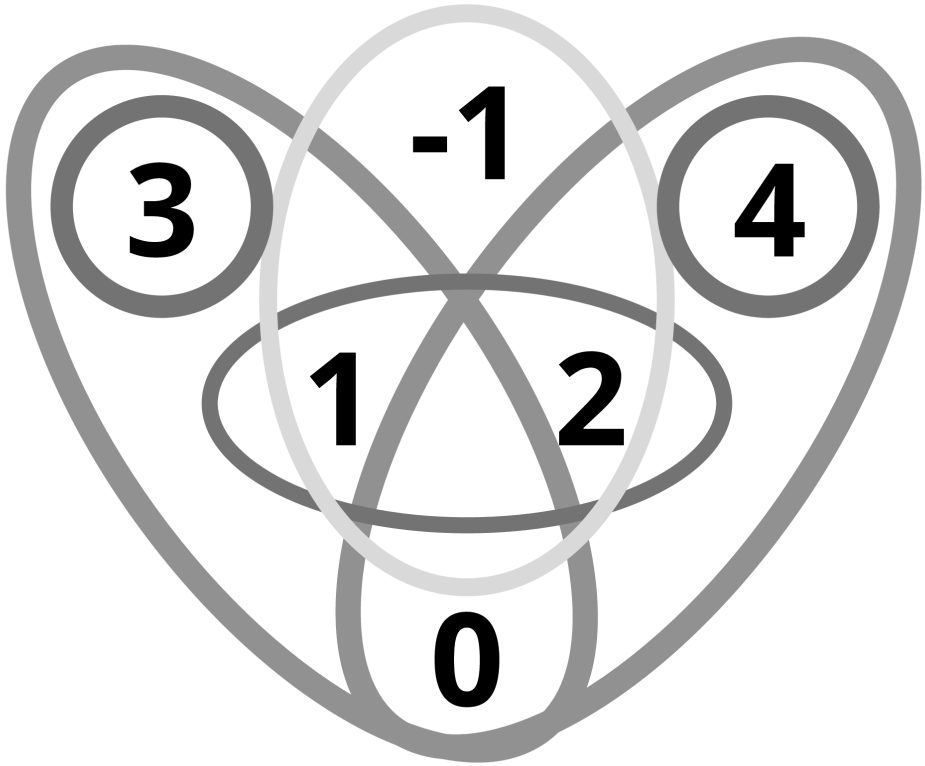
(Image credit: Wikipedia.)

Intertwined points



Q. Which points here are intertwined?

Intertwined points

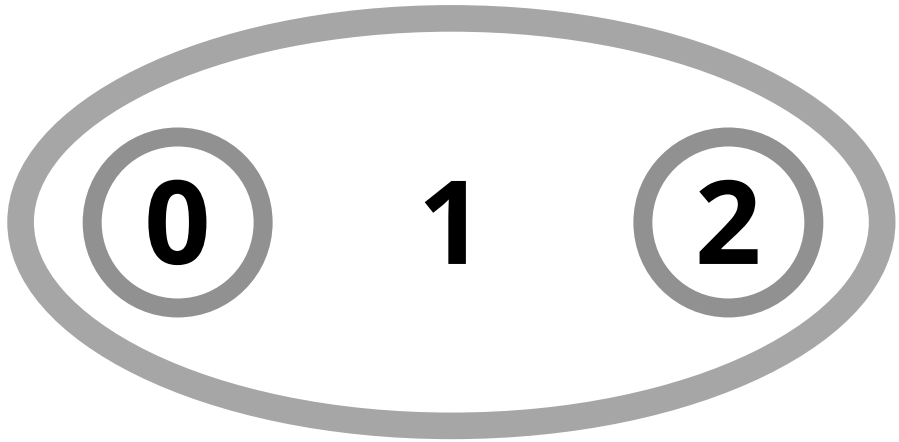


Q. Which points here are intertwined?

A. -1 0 1 2

Enter paraconsistent logic

Consider a continuous function *val* from this semitopology



to Bool $= \{\mathbf{t}, \mathbf{f}\}$ with the discrete (semi)topology.

If $val(0) = \mathbf{t}$ and val$(2) = \mathbf{f}$, then no value for $val(1)$ can make $val^{-1}(\{\mathbf{t}\})$ and $val^{-1}(\{\mathbf{f}\})$ both be open.

We need a third value **b** for 'both'.

Paraconsistent logic

Definition. Define $\mathbf{3} = \{\mathbf{t}, \mathbf{b}, \mathbf{f}\}$, with semitopology

$$\begin{aligned} \text{Open} &= \{\varnothing, \{\mathbf{t}\}, \{\mathbf{f}\}, \{\mathbf{t}, \mathbf{f}\}, \mathbf{3}\} \\ \text{Closed} &= \{\mathbf{3}, \{\mathbf{t}, \mathbf{b}\}, \{\mathbf{f}, \mathbf{b}\}, \{\mathbf{b}\}, \varnothing\}. \end{aligned}$$

Make $\mathbf{3}$ into a lattice, ordered as $\mathbf{t} \geq \mathbf{b} \geq \mathbf{f}$, with operators

- $\wedge$ (glb),
- $\vee$ (lub),
- and $\neg$ (reverses the lattice, thus swaps $\mathbf{t}$ and $\mathbf{f}$ and fixes $\mathbf{b}$).

From this we can build a material implication $\rightsquigarrow$ and iff $\leftrightsquigarrow$.

Going back to the previous example, we can safely set $val(1) = \mathbf{b}$ and the valuation becomes continuous.

Some truth tables

$\neg p$	
t	**f**
b	**b**
f	**t**

$p \wedge q$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**b**	**b**	**f**
f	**f**	**f**	**f**

$p \vee q$	**t**	**b**	**f**
t	**t**	**t**	**t**
b	**t**	**b**	**b**
f	**t**	**b**	**f**

$(\neg p) \vee q = p \rightsquigarrow q$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**t**	**b**	$\underline{\mathbf{b}}$
f	**t**	**t**	**t**

$p \leftrightsquigarrow q = p \rightsquigarrow q \wedge q \rightsquigarrow p$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**b**	**b**	**b**
f	**f**	**b**	**t**

$p \Rightarrow q$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**t**	**b**	$\underline{\mathbf{f}}$
f	**t**	**t**	**t**

$p \Leftrightarrow q$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**b**	**b**	**f**
f	**f**	**f**	**t**

Using logic to assert semitopological properties

Claims.

1. Continuity of a valuation $val : S \to \mathbf{3}$ can be expressed as a Horn clause theory in $\Rightarrow$, where $p \in S$ are propositinal atoms.
2. Any semitopology S can be represented as a Horn clause theory $Horn(S)$. Closed sets are answer sets of $Horn(S)$ — makes sense intuitively: intersections of answer sets to Horn clauses, are answer sets.
3. The assertion 'p is intertwined with q in S' corresponds to the judgement

$$Horn(S) \models p \leftrightsquigarrow q$$

   (validity $\models$ defined on next slide).
4. A sequent system (not shown) exists for derivability in $\mathbf{3}$, allowing assertions to be checked using a standard sequent derivation search.

The knowledge modality

I won't have time to go into detail, but for more advanced properties we also want a knowledge modality K. I want to show you this, because I've not seen a paraconsistent logic with a knowledge modality before!

$$
\begin{aligned}
[x]_{val} &= x & x &\in \mathbf{3} \\
[p]_{val} &= val(p) & p &\in \mathsf{P} \\
[\mathsf{Q}\,\phi]_{val} &= \mathsf{Q}[\phi]_f & \mathsf{Q} &\in \{\neg, \dots\} \\
[\phi\,\mathsf{Q}\,\phi']_{val} &= [\phi]_f \circ [\phi']_f & \mathsf{Q} &\in \{\wedge, \vee, \rightsquigarrow, \Rightarrow\}
\end{aligned}
$$

$$[\mathsf{K}(\phi)]_{val} = \bigwedge\{[\phi]_{val} \mid val : \mathsf{P} \to \mathbf{3}\}$$

$\Phi \models \Psi$ when for every val,

$$\Big(\bigwedge_{\phi\in\Phi} [\phi]_{\mathsf{val}} \rightsquigarrow \bigvee_{\psi\in\Psi} [\psi]_{\mathsf{val}}\Big) \in \{\mathbf{t}, \mathbf{b}\}.$$

Conclusions

1. Introducing semitopologies.
2. Intimate connection between semitopologies and Horn clause theories over a simple paraconsistent logic.
3. (Anti)connectivity properties correspond to simple logical assertions.
4. Verification of safety properties for real distributed networks suggest interesting directions in the logic.
5. More on semitopologies in this journal draft: https://arxiv.org/abs/2303.09287.